

**Exotic resonant level models in non-Abelian quantum Hall states coupled to quantum dots**Gregory A. Fiete,<sup>1</sup> Waheb Bishara,<sup>2</sup> and Chetan Nayak<sup>3,4</sup><sup>1</sup>*Department of Physics, The University of Texas at Austin, Austin, Texas 78712, USA*<sup>2</sup>*Department of Physics, California Institute of Technology, MC 114-36, Pasadena, California 91125, USA*<sup>3</sup>*Microsoft Research, Station Q, CNSI Building, University of California, Santa Barbara, California 93106, USA*<sup>4</sup>*Department of Physics, University of California, Santa Barbara, California 93106, USA*

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In this paper, we study the coupling between a quantum dot and the edge of a non-Abelian fractional quantum Hall state. We assume the dot is small enough that its level spacing is large compared to both the temperature and the coupling to the spatially proximate bulk non-Abelian fractional quantum Hall state. We focus on the physics of level degeneracy with electron number on the dot. The physics of such a resonant level is governed by a  $k$ -channel Kondo model when the quantum Hall state is a Read-Rezayi state at filling fraction  $\nu=2+k/(k+2)$  or its particle-hole conjugate at  $\nu=2+2/(k+2)$ . The  $k$ -channel Kondo model is channel symmetric even without fine tuning any couplings in the former state; in the latter, it is generically channel asymmetric. The two limits exhibit non-Fermi-liquid and Fermi-liquid properties, respectively, and therefore may be distinguished. By exploiting the mapping between the resonant level model and the multichannel Kondo model, we discuss the thermodynamic and transport properties of the system. In the special case of  $k=2$ , our results provide a distinct venue to distinguish between the Pfaffian and anti-Pfaffian states at filling fraction  $\nu=5/2$ . We present numerical estimates for realizing this scenario in experiment.

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**I. INTRODUCTION**

One of the most outstanding problems in the fractional quantum Hall effect is the unambiguous experimental identification of a state with non-Abelian quasiparticle braiding statistics.<sup>1</sup> To date, there is no physical system that has been conclusively demonstrated to have this property. Certain fractional quantum Hall states, most notably  $\nu=5/2$  and  $\nu=12/5$ , are believed to be exceptionally promising candidates for possessing non-Abelian quasiparticles and their experimental verification would constitute a major milestone in physics. The evidence that the  $\nu=5/2$  quantum Hall state is non-Abelian comes from a combination of analytical theory;<sup>2-4</sup> numerical study of small systems;<sup>5-10</sup> and, most importantly, experiment itself.<sup>11-13</sup> In the case of the  $\nu=12/5$  state,<sup>14,15</sup> conjectures of non-Abelian-ness are based solely on numerics.<sup>8,10,16</sup>

Non-Abelian quantum Hall states were first constructed with the use of conformal field theory (CFT).<sup>17</sup> Moore and Read (MR) (Ref. 2) argued that candidate wave functions could be computed as certain correlation functions (conformal blocks) of a CFT. By requiring that the wave functions have certain physical properties, the set of allowed CFTs turn out to be highly constrained. The simplest CFT that satisfies the constraints and possess non-Abelian quasiparticles is the  $\mathbb{Z}_2$  Ising CFT which leads to a wave function known as the MR ‘‘Pfaffian’’ state so named because the wave function contains a Pfaffian factor. Numerics<sup>5,6,10,18</sup> supports this wave function as a candidate description of the  $\nu=5/2$  plateau,<sup>19,20</sup> which has the largest gap among observed plateaus in the second Landau level.<sup>21-23</sup>

The hypothesis that the  $\nu=5/2$  state is non-Abelian has, until recently, been based on the MR wave function<sup>2</sup> and quasiparticle excitations above it.<sup>4</sup> However, it was recently realized that there is an alternate non-Abelian candidate for

$\nu=5/2$ , the *anti-Pfaffian* state<sup>24,25</sup> which, in the limit of vanishing Landau-level mixing, is exactly degenerate in energy with the MR Pfaffian state (although both might, in principle, be higher in energy than some other state). Exact diagonalization studies have generally neglected Landau-level mixing and have, therefore, not distinguished between these two states. Early numerics on the sphere<sup>5</sup> failed to look for the anti-Pfaffian state, which occurs at a different shift. On the torus,<sup>6</sup> finite-size effects, which cause mixing between these two states and a consequent energy splitting, were too large. However, more recent numerics on the sphere<sup>18</sup> and the torus<sup>26</sup> make it clear that, in the absence of Landau-level mixing, the anti-Pfaffian state is as good a candidate as the MR Pfaffian state with a ground-state energy which converges with that of the MR Pfaffian in the thermodynamic limit.

Recent work on the effects of Landau-level mixing indicate that the anti-Pfaffian state is, in fact, lower in energy for realistic values of the magnetic field.<sup>7,27</sup> However, it is difficult to experimentally distinguish the anti-Pfaffian state from the MR Pfaffian state. The smallest quasiparticle charge is  $e/4$  in both states, so shot-noise measurements<sup>11</sup> cannot distinguish between them. The two states’ non-Abelian braiding properties are very similar. Thus, interferometry experiments,<sup>27-34</sup> such as those of Willett *et al.*<sup>13</sup> are also unlikely to distinguish between these two states. It is possible, in principle, to distinguish them by their  $I(V, T)$  curves, but these can be modified by edge reconstruction, so the distinction between the  $I(V, T)$  curves of the two states might be blurred. The two states can also be distinguished by the signs of their thermal Hall conductances but only if the larger contribution of the filled lowest Landau level (of both spins) can be separated. Therefore, there is a need for experiments capable of distinguishing these two candidate descriptions of the  $\nu=5/2$  state. In this paper, we show how differences in the gapless edge modes of the anti-Pfaffian and MR Pfaffian

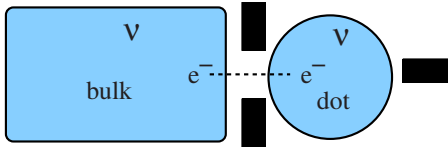


FIG. 1. (Color online) Schematic of our model. Gates are shown in black. They may be used to form a point contact to pinch off the dot from the rest of the quantum Hall bulk. The gate on the right of the figure may be used to shift the energy levels of the dot by changing its area  $S$ . The bulk is assumed to be at filling fraction  $\nu=2+k/(k+2)$  or  $\nu=2+2/(k+2)$ . The white region between the dot and the bulk is assumed to be at  $\nu=2$ . The charge on the dot may be measured capacitively (Refs. 40 and 41).

states lead to different quantum impurity (resonant level) models and transport properties when these states are coupled to quantum dots.<sup>35</sup> We also briefly discuss the expectations for other (numerically less competitive) quantum Hall states at  $\nu=5/2$ , such as the  $SU(2)_2$  NAF,<sup>36,37</sup>  $K=8$  strong pairing,<sup>38</sup> and  $(3,3,1)$  state.<sup>38</sup>

The MR Pfaffian state was generalized by Read and Rezayi<sup>14</sup> (RR), who constructed wave functions using the  $Z_k$  parafermion CFTs (the case  $k=2$  is the MR Pfaffian wave function). These wave functions are candidate states for filling fraction  $\nu=2+k/(k+2)$  and they all possess non-Abelian quasiparticle braiding statistics for  $k \geq 2$ . The particle-hole conjugates of the RR states, which we will call the “anti-RR states,” were constructed recently;<sup>39</sup> the  $k=3$  anti-RR state is a candidate description of the  $\nu=12/5$  plateau. Another class of non-Abelian candidate states follow from Bonderson and Slingerland’s<sup>15</sup> (BS) hierarchy construction built on the  $\nu=5/2$  state. At  $\nu=12/5$ , the  $k=3$  anti-RR state and one of the BS states are distinct non-Abelian candidates. Our setup can be applied to distinguish these two states.

A subset of the results presented here was succinctly given in Ref. 35. The basic scenario we discuss is shown in Fig. 1. Our main result is that electron tunneling between a level- $k$  RR state and a quantum dot at its degeneracy point,  $E(N+1)=E(N)$ , can be mapped to a *stable*  $k$ -channel, channel symmetric, Kondo model [where  $k \geq 2$  is given by the filling fraction  $\nu=2+k/(k+2)$ , assumed to be described by a RR state]. Here  $E(N)$  is the energy of the quantum dot with  $N$  electrons on it. Since the non-Fermi-liquid physics of multichannel Kondo models with channel isotropy is usually unstable to small anisotropies in the coupling to different channels, this physics is very difficult to realize in experiment. A central result of this work is that the non-Abelian nature of the RR fractional quantum Hall states *automatically* provides channel isotropy in the emergent multichannel Kondo model. In this sense, the non-Abelian fractional quantum Hall state provides “topological protection” to the non-Fermi-liquid physics of the multichannel Kondo model. In this way, we have uncovered a scenario in which a global property, the topological order of the RR states, protects fragile *local* quantum impurity physics. Our analysis is based on properties of the gapless edge excitations required by the topological order of the RR states.

We also consider the situation in which two bulk quantum Hall droplets are separated by a quantum dot (i.e., another

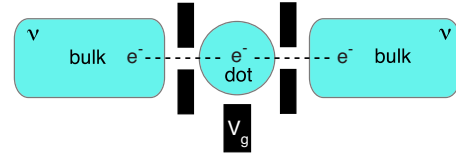


FIG. 2. (Color online) Schematic of our model for electron tunneling through a quantum dot. Gates are shown in black. They may be used to form a point contact to pinch off the dot from the rest of the quantum Hall bulk. The gate on the bottom of the figure may be used to shift the energy levels of the dot by changing its area  $S$  through a change in the gate voltage  $V_g$ . The bulk is assumed to be at filling fraction  $\nu=2+k/(k+2)$  or  $\nu=2+2/(k+2)$ . The white region between the dot and the bulk is assumed to be at  $\nu=2$ .

bulk droplet on the right side of the dot in Fig. 1, as shown in Fig. 2). This setup is relevant to a two point-contact interferometer<sup>13,28–30,42,43</sup> when the backscattering at each point contact is large so that it is in the Coulomb-blockade regime.<sup>44</sup> One may naively think that this scenario would “double” the number of “conduction” electron channels and lead to a  $2k$ -channel Kondo model. However, we show explicitly that this is not the case and that its true fixed point (for equal electron tunneling to both “bulk” states) is more subtle.

This paper is organized as follows. In Sec. II, we introduce the essential elements of the edge theory of Abelian and non-Abelian fractional quantum Hall edge states, including the particle-hole conjugate states. In Sec. III, we review the Emery-Kivelson (EK) solution to the two-channel Kondo model and the more general Affleck-Ludwig CFT solution to the  $k$ -channel Kondo model. The CFT solution of this problem bears a deep connection to the RR states. In Sec. IV, we describe the physics of a resonant level tunnel coupled to the edge of a RR state and map it to the  $k$ -channel Kondo model using the results of Sec. III. We also discuss the tunneling of an electron *through* the dot. Finally, in Sec. V we remark on the experimental implementation of our scenario and summarize in Sec. VI.

## II. EDGE THEORIES OF QUANTUM HALL STATES

Fractional quantum Hall states are topologically ordered<sup>45</sup> states of matter: They are gapped in the bulk and contain quasiparticle excitations with nontrivial charge and braiding statistics. Broken time-reversal symmetry in quantum Hall systems leads to net flow of charge in one direction along the boundary. In the absence of edge reconstruction,<sup>46–50</sup> the number of independent edge excitations and their low-energy properties are determined by the topological order of the bulk state.<sup>45</sup> In this way, there is a “bulk-edge” correspondence in quantum Hall systems.<sup>1,51</sup> CFT can provide a precise description of this correspondence but note Refs. 52 and 53. A given CFT can be used to produce trial bulk wave functions in two spatial dimensions with and without quasiparticles.<sup>2,4,14,54–56</sup> The same CFT can also be used to describe the dynamics of edge excitations in 1+1 dimensions. Of course, whether a particular CFT is chosen by nature ultimately depends on whether the corresponding state

has a lower energy than other candidate states. Fortunately, some of the interesting CFTs (namely, the  $Z_k$  parafermion theories) do seem to be favored or at least very competitive at certain filling fractions (typically for  $2 < \nu < 3$ ) and with reasonable interactions.<sup>7,14,16</sup>

Our analysis of the coupling between the edge of a non-Abelian fractional quantum Hall state and a quantum dot relies heavily on the theory of the edge. We focus on the edges of the RR states and their particle-hole conjugates for general  $k \geq 2$ , which contain the physics of the chiral  $Z_k$  parafermion CFT (Ref. 17) in addition to the physics of the Abelian quantum Hall states. In this section, we present a self-contained summary of the most important aspects of the edge needed for our study. In order to begin our narrative on familiar ground, we start the discussion by reviewing the essential properties of the edge states of the more widely known Abelian quantum Hall states. In order to simplify our presentation, we will assume throughout this paper that edge reconstruction does not occur.<sup>47-50</sup> We will discuss the qualitative effects of edge reconstruction in Sec. V.

### A. Laughlin and Hierarchy edge states

In the integer quantum Hall effect, where electron-electron interactions can be neglected for many purposes, gapless edge excitations occur whenever a Landau level is pushed through the Fermi energy near a boundary.<sup>57</sup> A quantum Hall state of integer filling  $\nu = n$  has  $n$  chiral one-dimensional (1D) (Dirac) fermionic modes that propagate along the boundary. These chiral modes carry heat and charge along the edge. A single chiral Dirac fermion has central charge  $c=1$  so for filling fraction  $n$ , the total effective central charge is  $c=n$ . Since the Hall conductivity,  $\sigma_H = n \frac{e^2}{h}$ , and the thermal conductivity,  $\kappa_H = n \frac{\pi^2 k_B^2}{3h} T$ , in dimensionless units the Hall conductivity and the thermal Hall conductivity are equal. Here  $e$  is the charge of the electron,  $h$  Planck's constant,  $k_B$  Boltzmann's constant, and  $T$  the temperature.

Qualitatively, the situation is similar in an Abelian fractional quantum Hall state: The bulk is gapped and the edge contains an integer number of chiral gapless edge modes. However, due to the central role that interactions play in the fractional quantum Hall effect, these edge modes carry charges which are a fraction of the electron charge. The simplest example is the Laughlin state at filling fraction  $\nu = 1/m$ , with  $m$  an odd integer. A Laughlin state contains a single edge mode. Wen demonstrated<sup>58,59</sup> that the universal properties of this edge mode are described by a chiral Luttinger liquid theory (a Gaussian CFT) with edge mode charges and correlations determined by  $\nu$  ( $\nu=1$  is a special case of this theory). Since a Gaussian CFT has  $c=1$ , and the Laughlin state has only one edge mode, the total central charge of the edge is  $c=1$ . It should be recalled that while the Hall conductance depends on the value of the edge charge,<sup>60</sup> the thermal Hall conductance does not: all Laughlin states and the  $\nu=1$  integer quantum Hall state have identical  $\kappa_H$ . Meanwhile,  $\sigma_H = \nu \frac{e^2}{h}$  differs for these states.

The low-energy edge theory of the Laughlin state (and  $\nu=1$ ) can be expressed in terms of bosonic fields and has an action that takes the form

$$S_{\text{edge}}^{\text{Laugh}} = \frac{1}{4\pi\nu} \int d\tau dx \partial_x \phi (i\partial_\tau + v\partial_x) \phi, \quad (1)$$

where the chiral bosonic fields satisfy the commutation relations  $[\phi(x'), \phi(x)] = i\pi\nu \text{sgn}(x-x')$ ; and  $v$  is the velocity of the edge mode, determined by nonuniversal properties of the edge confining potential and interactions. (Throughout our paper we have set  $\hbar=1$ , which multiplies the velocity  $v$  here and in most places in the text.) The electron operator on the edge is

$$\Psi_{e,\text{Laugh}}^\dagger = e^{i\phi/\nu} \quad (2)$$

and the quasiparticle operator of charge  $\nu e$  is  $\Psi_{qp,\text{Laugh}}^\dagger = e^{i\phi}$ .

We also argued<sup>61-63</sup> that the same structure appears in a general hierarchy state<sup>64,65</sup> but with multiple chiral edge modes.<sup>58,66</sup> This more general theory of the Abelian quantum Hall hierarchy states takes the form

$$S_{\text{edge}}^{\text{Hier}} = \frac{1}{4\pi} \int d\tau dx [i\partial_\tau \phi_i K_{ij} \partial_x \phi_j + \partial_x \phi_i V_{ij} \partial_x \phi_j + 2A_{\mu i} t_i \epsilon_{\mu\nu} \partial_\nu \phi_i]. \quad (3)$$

The  $n \times n$  matrix  $K_{ij}$  and the  $n$ -component vector  $t_i$  encode the topological properties of the quantum Hall state. For instance,  $K_{ij} = \delta_{ij} + 2p$ , when the filling fraction is of the form  $\nu = \frac{n}{2pn+1}$  (the Jain sequence<sup>67</sup>) with  $n$  and  $p$  integers. The  $n \times n$  matrix  $V_{ij}$  describes the nonuniversal velocities and interactions between the  $n$  edge modes. An arbitrary excitation on the edge is created by an operator  $\hat{T}_{\{m_j\}}(x) = e^{i\sum_{j=1}^n m_j \phi_j(x)}$ , where  $m_j$  are arbitrary integers.<sup>62</sup> The topological properties of the state are (i) ground-state degeneracy on surface of genus  $g$  is  $\text{Det}(K)^g$  where  $\text{Det}$  is the determinant. (ii) Filling fraction  $\nu = \sum_{ij} t_i K_{ij}^{-1} t_j$ . (iii) Statistical angle of a particle created by  $\hat{T}_{\{m_j\}}$  is  $\frac{\theta}{\pi} = \sum_{ij} m_i K_{ij}^{-1} m_j$ . (iv) Charge of a particle created by  $\hat{T}_{\{m_j\}}$   $Q = \sum_{ij} m_i K_{ij}^{-1} t_j$ . Note that both  $K_{ij}$  and  $t_i$  are basis dependent, and an  $SL(n, \mathbb{Z})$  transformation can transform them between different bases. In the "symmetric basis," in which  $t_i = 1$  for all  $i$ , the electron operator takes the form

$$\Psi_{e,\text{Hier}}^\dagger = e^{i\sum_{j=1}^n m_j \phi_j(x)}, \quad (4)$$

where  $m_i = \sum_j K_{ij} l_j$ ,  $\sum_i l_i = 1$ , and  $l_j = \sum_i m_i K_{ij}^{-1}$ . This ensures the electron has a statistical angle of  $\pi$  (mod  $2\pi$ ) and a unit charge. It can readily be checked that the formulas above reduce to the correct expressions in a Laughlin state where  $K_{ij} \rightarrow 1/\nu = m$  and there is only one field  $\phi$  in the edge theory. In the hierarchy states,  $\kappa_H = n \frac{\pi^2 k_B^2}{3h} T$  when all  $n$  edge modes move in the same direction. Otherwise,  $\kappa_H = (c_-) \frac{\pi^2 k_B^2}{3h} T$ , with  $c_-$  equal to the difference of the central charges of the right- and left-moving edge modes. If  $c_- < 0$ , heat flows in the direction opposite to the charge. This happens, for example, in the  $\nu=3/5$  state.

A few remarks are in order regarding the edge theories of Abelian fractional quantum Hall states. First, we note that all edge channels in the theory [Eq. (3)] carry heat in the theory described above. In the symmetric basis mentioned above, all edge channels carry charge as well. (Depending on the

nonuniversal matrix  $V_{ij}$ , the symmetric basis or, perhaps, another basis may be the eigenmodes of the Hamiltonian.) The edge theories of non-Abelian fractional quantum Hall states always contain additional electrically neutral modes that carry no charge but do carry heat. These neutral modes are responsible for the non-Abelian properties of the quantum Hall state. In the class of quantum Hall states, we consider in this paper, the neutral modes will correspond to either Majorana fermions, or their generalization, parafermions (to be described below).

Second, disorder at the edge plays an important role for fractional quantum Hall states with counterpropagating edge modes.<sup>58,66</sup> If all the charged modes are propagating in the same direction, disorder has no effect at the boundary because edge modes have no states into which they can backscatter. Forward scattering is unimportant as it has no qualitative effect on the physics. On the other hand, if the edge contains counterpropagating modes it is possible for backscattering to occur in the presence of disorder. As is well known, backscattering in one-dimensional interacting systems can lead to dramatic effects.<sup>68</sup> One can then analyze the disorder terms within a renormalization group approach. A new effective edge theory can emerge at low energies when disorder is relevant.<sup>69,70</sup> Whether a given state in the hierarchy has counterpropagating modes can be determined simply by looking at the sign of the eigenvalues of the matrix  $K$  appearing in Eq. (3). If there are eigenvalues of opposite sign, there are counterpropagating modes. This situation occurs at filling fraction  $\nu=2/3$  for which  $n=2$  and  $p=-1$ , for example.<sup>58,66</sup> As a rule of thumb, particle-hole conjugate states contain counterpropagating edge modes and one needs to consider the effect of edge disorder. (The  $\nu=2/3$  state can be thought of as  $\nu=1/3$  fluid of holes in a  $\nu=1$  background. Thus, the edge theory in the absence of disorder can be described by an “outer”  $\nu=1$  edge and an “inner”  $\nu=-1/3$  counter propagating edge.) The same scenario will arise in particle-hole conjugate states in non-Abelian quantum Hall systems. One must analyze the relevance of disorder there as well and determine the new effective low-energy theory of the edge if disorder is relevant.<sup>24,25,39</sup>

### B. Moore-Read Pfaffian edge state

Fermi statistics requires that a large class of Abelian states have odd-denominator filling fractions, such as the most prominently observed ones, which have  $\nu=\frac{n}{2pn+1}$ .<sup>67</sup> It thus came as a surprise when a Hall plateau at filling fraction  $\nu=5/2$  was observed.<sup>19</sup> Fractional quantum Hall states with even-denominator filling fractions were proposed, based on the idea that electrons could form bosonic pairs which could then condense into a bosonic analogue of the Laughlin state.<sup>2,3,38,71</sup> Such states could be either Abelian<sup>38</sup> or non-Abelian.<sup>2</sup> (It was later realized that one proposed state<sup>71</sup> was actually critical.<sup>72</sup>) These states are distinguished by the topological properties of their pair wave functions. The MR Pfaffian<sup>2</sup> state is characterized by a weakly bound<sup>72</sup>  $p_x+ip_y$  pair wave function. (The anti-Pfaffian state is characterized by a weakly bound  $p_x-ip_y$  pair wave function of holes, or equivalently a  $p_x-ip_y$  state of electrons.<sup>24,25</sup>) The MR Pfaff-

ian wave function was constructed by adding an electrically neutral chiral  $\mathbb{Z}_2$  Ising CFT to the  $c=1$  chiral boson CFT that describes the edge of the Laughlin states. (One typically assumes that the filled lower  $\nu=2$  Landau levels are inert and play no role other than to perhaps modify the effective electron-electron interactions so that pairing is energetically favorable.) The MR state, like the Laughlin states, is spin polarized. Its edge action is the sum of the neutral and charged sectors,  $S=S_n+S_c$  with

$$S_n^{\text{MR}} = \int d\tau dx \ i\psi(i\partial_\tau\psi + v_n\partial_x\psi) \quad (5)$$

and

$$S_c = \frac{2}{4\pi} \int d\tau dx \partial_x\phi(i\partial_\tau\phi + v_c\partial_x\phi), \quad (6)$$

where  $v_n$  is the neutral mode velocity and  $v_c$  is the charge mode velocity. Typically  $v_n < v_c$ , because of the repulsive interactions in the charge sector.<sup>73,74</sup> Note that Eq. (6) is just Eq. (1) with  $\nu=1/2$  and  $v=v_c$ .

Equation (5) is equivalent to the right-moving part of the quantum critical  $1+1$ -D transverse field Ising model [or the classical critical two-dimensional (2D) Ising model].  $\psi$  is a Majorana fermion operator. In the Ising model, it creates a domain wall between up and down spins. The Ising model spin field  $\sigma$  is a twist field which creates a branch cut in  $\psi$ . The field  $\sigma$  is important for understanding non-Abelian statistics in the MR Pfaffian state and for charge  $e/4$  quasiparticle tunneling from one edge of a droplet to the other. However, we will not be using it in this paper because we focus here on electron tunneling between one droplet and another much smaller droplet (i.e., the quantum dot).

The electron creation operator in the MR state is

$$\Psi_{e,\text{MR}}^\dagger = \psi e^{i2\phi}, \quad (7)$$

where we have used the convention  $\text{dim}[e^{i\alpha\phi}] = \nu\frac{\alpha^2}{2}$  and  $\text{dim}[\psi] = 1/2$  so that  $\text{dim}[\Psi_{e,\text{MR}}] = 3/2$ . Note that the electron operator [Eq. (7)] is a combination of a Majorana fermion and an exponential of the boson, identical to that appearing in Eq. (2). The appearance of a *single* Majorana fermion in the electron operator is the crucial factor that allows the mapping of electron tunneling to and from a resonant level to the channel symmetric two-channel Kondo model.<sup>35</sup> Since a Majorana fermion has central charge  $1/2$  and, in the MR Pfaffian state, moves in the same direction as the charge mode, the total central charge of the edge is  $3/2$ , which implies  $\kappa_H = \frac{3}{2} \frac{\pi^2 k_B^2}{3h} T$ .

### C. Read-Rezayi edge states

The Majorana fermion appearing in the theory of the MR state is just the  $k=2$  special case of  $\mathbb{Z}_k$  parafermions, which invites generalizations to higher  $k$ . These were constructed by Read and Rezayi;<sup>14</sup> they are spin-polarized states with filling fraction  $\nu=k/(k+2)$ . They may be applicable to observed states at  $\nu=2+k/(k+2)$  if one, again, assumes that the filled lower Landau levels are inert.

The  $\mathbb{Z}_k$  parafermion theories arise as the self-dual critical points of  $\mathbb{Z}_k$  clock models.<sup>75</sup> In contrast to the chiral  $\mathbb{Z}_2$  Ma-

fermions, the  $\mathbb{Z}_k$  parafermions<sup>76</sup> for general  $k > 2$  are not free theories.<sup>17</sup> They have central charge  $c = \frac{2k-2}{k+2}$  and can be realized as  $SU(2)_k/U(1)$  cosets. This coset can be realized at the Lagrangian level by an  $SU(2)_k$  chiral WZW model in which the  $U(1)$  subgroup has been gauged,<sup>77</sup>

$$S_{SU(2)_k/U(1)} = \frac{k}{16\pi} \int d\tau dx \operatorname{tr}(\partial_x g^{-1} \bar{\partial} g) - i \frac{k}{24\pi} \int dxd\tau dr \epsilon^{\mu\nu\lambda} \operatorname{tr}(\partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\lambda g g^{-1}) + \frac{k}{4\pi} \int dxd\tau \operatorname{tr}(A_x \bar{\partial} g \cdot g^{-1} - \bar{A} g^{-1} \partial_x g + A_x g \bar{A} g^{-1} - A_x \bar{A}). \quad (8)$$

In this expression,  $\bar{\partial} \equiv i\partial_\tau + v_c \partial_x$  and  $\bar{A} \equiv A_\tau - iv_n A_x$ . The field  $g$  takes values in  $SU(2)$ . The second integral is over any three-dimensional manifold  $M$  which is bounded by the two-dimensional space time of the edge  $\partial M$ . The value of this integral depends only on the values of the field  $g$  at the boundary  $\partial M$ . The gauge field has no Maxwell term so its effect is to set to zero the  $U(1)$  current to which it is coupled.

The coset construction can be realized at the operator level by writing the  $SU(2)_k$  currents in terms of a chiral boson  $\phi$  and a parafermion field  $\psi_1$ ,

$$J^+ = \sqrt{k} \psi_1 e^{i\phi}, \\ J^- = \sqrt{k} \psi_1^\dagger e^{-i\phi}, \\ J^z = \frac{k}{2} \partial_x \phi, \quad (9)$$

where the chiral boson is normalized so that  $\dim[e^{i\phi}] = 1/k$ . The parafermion field is what is left over after the chiral boson has been “removed” from the  $SU(2)$  current. Throughout this paper, different normalization of the bosonic fields will be used as a consequence of the different charge sector Hamiltonians at different filling fractions but the scaling dimension of the  $SU(2)$  currents will always remain unity. There are also other primary fields in the  $\mathbb{Z}_k$  parafermion theory, such as powers of the spin field. Although they are important for *quasiparticle* tunneling from one edge of a droplet to the other through the bulk, they do not enter our analysis here for electrons, where the parafermion field is central.

The charge sector of the RR edge theory (neglecting the two filled lower Landau levels) is identical to Eq. (1) with  $\nu = k/(k+2)$ . The electron operator in the RR state is obtained by combining appropriate fields from the charge and neutral sectors,<sup>14</sup>

$$\Psi_{e,RR}^\dagger = \psi_1 e^{i(k+2/k)\phi}, \quad (10)$$

which has  $\dim[\Psi_{e,RR}] = 3/2$ , independent of  $k$ . With the  $c = 1$  charge sector included, the total central charge of the edge is  $\frac{3k}{k+2}$ , which implies  $\kappa_H = \frac{3k}{k+2} \frac{\pi^2 k_B^2}{3h} T$  in the RR states.

#### D. Particle-hole conjugates of MR and RR states

Particle-hole conjugate fractional quantum Hall states can be thought of as a Hall fluid of holes in a background of electrons at integer filling.<sup>58,66</sup> This picture naturally leads to an “outer edge” with properties like that of an integer quantum Hall state, and an “inner edge” (that propagates in the opposite direction because of the opposite charge of the holes relative to electrons) that is at a filling  $1 - \nu$ , where  $0 < \nu < 1$  is the filling fraction of the state that is conjugated. (We have assumed the particle-hole conjugation is in the lowest Landau level by again neglecting the lower two filled Landau levels. If the filling fraction is in the range  $2 < \nu < 3$ , then the particle-hole conjugate state would be at filling fraction  $5 - \nu$ .) As noted in Sec. I, the MR Pfaffian state  $\nu = 5/2$  has a particle-hole conjugate state (degenerate in the absence of Landau level mixing, which is a particle-hole symmetry breaking perturbation) that is at the same filling fraction.<sup>24,25</sup> This immediately raises an additional question: “is the MR Pfaffian state or its particle-hole conjugate a better candidate for  $\nu = 5/2$ ?” Our work provides an experimental test for this question quite distinct from existing proposals.

As we mentioned in our discussion of the hierarchy states, the edge physics is more subtle for states that have counter-propagating modes on the edge. Much of the nontrivial physics of one dimension is related to backscattering in the presence of interactions. Counterpropagating modes allow for such backscattering, provided the disorder causing the backscattering has Fourier components at the right values to soak up the momentum mismatch between the two (or more) modes. The relevant analysis in the fractional quantum Hall context was first carried out by Kane *et al.* for the  $\nu = 2/3$  state.<sup>69</sup> The key result is that the edge has two phases: one in which the disorder is a relevant perturbation to the “clean” edge theory and one in which it is not. When the disorder is irrelevant, the picture of an outer edge with properties of an integer quantum Hall state and an inner counterpropagating edge with properties of a fractional quantum Hall state applies. On the other hand, when disorder is relevant [which, in turn, depends on the nonuniversal matrix  $V_{ij}$  in Eq. (3)], the edge reconstructs at long length scales into an electrically neutral (disorder dependent) mode and a Laughlin-type charge mode effectively at  $\nu = 2/3$ . A remarkable result is that the neutral mode has an emergent  $SU(2)$  symmetry.<sup>69</sup> This neutral mode propagates in the opposite direction to the charge mode.

Much of the analysis of Kane *et al.* for the  $\nu = 2/3$  state carries over to the particle-hole conjugates of the MR and RR states, although the technical details differ somewhat: the analysis for the non-Abelian states needs to be modified to take into account the neutral  $\mathbb{Z}_k$  chiral parafermion mode present in the inner edge.<sup>24,25,39</sup> When this analysis is carried out, the edge theory of the particle-hole conjugate of the MR state,  $\overline{\text{MR}}$ , is given by  $S = S_n^{\overline{\text{MR}}} + S_c$  with  $S_c$  given by Eq. (6) and a neutral sector of three flavors of Majorana fermions,

$$S_n^{\overline{\text{MR}}} = \sum_{a=1}^3 \int dxi \psi_a (-i\partial_\tau + v_n \partial_x) \psi_a, \quad (11)$$

that propagates in the direction opposite of the charge. Since the central charge of a Majorana fermion is  $1/2$  and there are

three flavors of them,  $\kappa_H = -\frac{1}{2} \frac{\pi^2 k_B^2}{3h} T$ , so that heat flows “upstream” relative to the current. (Of course, once the two lower Landau levels are taken into account, heat will still flow downstream along with the current but there will be a reduction relative to the MR state.) The different edge theories and thermal Hall conductivities for the MR and  $\overline{\text{MR}}$  states imply that these are topologically distinct states.<sup>24,25</sup> The neutral sector [Eq. (11)] is SU(2) symmetric and, in fact, the symmetry generators,

$$J^a = i \epsilon^{abc} \psi_b \psi_c, \quad (12)$$

obey an SU(2)<sub>2</sub> Kac-Moody algebra, i.e.,

$$J^a(z) \cdot J^b(0) = \frac{1}{z^2} \frac{k}{2} \delta^{ab} + \frac{1}{z} i \epsilon^{abc} J^c(0) + \dots \quad (13)$$

with  $k=2$ . By contrast, in the  $\nu=2/3$  case analyzed by Kane *et al.*,<sup>69</sup> the neutral sector can be formulated as an SU(2)<sub>1</sub> Kac-Moody algebra.

The electron operator in the  $\overline{\text{MR}}$  state is no longer unique as it was in the MR state and this will ultimately lead to different Kondo physics when the resonant level problem is mapped to the Kondo model. There are three different dimension-3/2 (the smallest possible scaling dimension) electron operators,<sup>24,25</sup>

$$(\Psi_{e,\overline{\text{MR}}}^\dagger)_a = \psi_a e^{2i\phi}, \quad (14)$$

for  $a=1,2,3$  where the combination  $(\psi_1 - i\psi_2) e^{2i\phi}$  is inherited from the electron operator of the  $\nu=1$  integer quantum Hall state in which a Pfaffian state of holes forms. The other two electron operators are complicated charge- $e$  combinations of the  $\nu=1$  electron operator and particle-hole excitations between the inner MR and outer  $\nu=1$  edges. The three flavors of Majorana fermions present in the MR state will turn out to drive the stable two-channel Kondo physics of the MR state to the less exotic single-channel Kondo physics in the MR state.<sup>35</sup>

The edge theory of the  $\overline{\text{RR}}$  state for general  $k \geq 2$  is a straightforward generalization of the MR state: the neutral sector is described by an SU(2) <sub>$k$</sub>  theory and the charge sector is given by Eq. (1) with  $\nu=2/(k+2)$ . The counterpropagating neutral sector is therefore given by a chiral WZW theory for SU(2) at level  $k$ ,

$$S_{\text{WZW},k} = \frac{k}{16} \int d\tau dx \text{tr}(\partial_x g^{-1} \bar{\partial} g) - i \frac{k}{24\pi} \int dr d\tau dx \epsilon^{\mu\nu\lambda} \text{tr}(\partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\lambda g g^{-1}). \quad (15)$$

The central charge of the neutral mode is  $\frac{3k}{k+2}$  so the net thermal conductivity is  $\kappa_H = -2 \left(\frac{k-1}{k+2}\right) \frac{\pi^2 k_B^2}{3h} T$ . Since  $k \geq 2$ , heat flows in the opposite direction of charge.

As in the case of the MR state, the electron operator is no longer unique.<sup>34,39</sup> There are  $2k+1$  electron operators with left- and right-scaling dimensions  $\Delta_R = (k+2)/4$ ,  $\Delta_L = k/4$ , so

that their total scaling dimension is  $(k+1)/2$  and their conformal spin is  $\Delta_R - \Delta_L = 1/2$ , as required for a fermion. These operators are given by<sup>39</sup>

$$(\Psi_{e,\overline{\text{RR}}}^\dagger)_m = \chi_{j=k/2}^m e^{i(k+2/2)\phi} \quad (16)$$

with  $m = -k/2, -k/2+1, \dots, k/2$ . The  $\chi_{j=k/2}^m$  are related to the spin  $j=k/2$  primary fields of SU(2) <sub>$k$</sub>  and are built entirely from fields in the neutral sector of the edge theory. The  $\chi_{j=k/2}^m$  have scaling dimension  $k/4$  and can be constructed by operating with the SU(2) <sub>$k$</sub>  current operator  $J^+ \sim \psi_1 e^{i\phi_\sigma}$  multiple times on a “bare outer edge” electron operator.<sup>39</sup> Here  $\phi_\sigma$  is an electrically neutral combination of the inner and outer charge edge modes<sup>39</sup> and  $\psi_1$  is the  $\mathbb{Z}_k$  parafermion. The nonuniqueness of the electron operator [Eq. (16)] will again turn out to drive the system away from the channel symmetric  $k$ -channel fixed point.

To summarize, the neutral sectors of the MR Pfaffian and RR edges are the Ising model (Majorana fermion) and  $\mathbb{Z}_k$ ,  $k \geq 3$  parafermion theories, respectively. These neutral sectors can be realized as the coset SU(2) <sub>$k$</sub> /U(1) ( $k=2$  is the Ising case). In contrast, the neutral sectors of the anti-Pfaffian and RR edges are oppositely directed SU(2) <sub>$k$</sub>  theories. In effect, particle-hole conjugation undoes the coset and restores the full SU(2) <sub>$k$</sub>  Kac-Moody algebra (in the phase in which disorder is relevant so that the edges equilibrate).

### III. $k$ -CHANNEL KONDO MODEL

In this section, we describe the  $k$ -channel Kondo model and its solution using analytical methods that we will encounter in the next section where our scenario of a quantum dot near a bulk non-Abelian quantum Hall state (see Fig. 1) is discussed. The most familiar guise of the Kondo model is as a local magnetic impurity (spin) in a sea of conduction electrons. Electrons scattering from this local moment can flip their spin. As a result, at low energies nontrivial correlations develop between the magnetic moment and the electrons of the many-body Fermi sea, and the local moment is screened (or perhaps “over” or “under” screened).<sup>78,79</sup> Describing this behavior in detail has been one of the most intensively studied problems in condensed-matter physics over the last 40 years.<sup>80</sup> Research on the problem is still quite active, particularly as it relates to finite frequency or non-equilibrium properties. The mapping we establish in this work may provide another venue to probe interesting non-equilibrium properties of a Kondo system, particularly in the multichannel case, which would be an interesting route for future study.

#### A. $k$ -channel Kondo Hamiltonian

Our results only require equilibrium properties which are well understood for the cases of interest to us. The most important distinction that needs to be made is between single-channel and multichannel versions of the Kondo model.<sup>78</sup> Here the “channels” refers to the number of “flavors” of electrons that couple to the impurity spin.

In the Kondo problem, one considers  $k$  channels of conduction electrons (e.g.,  $k$  bands) interacting with a localized

impurity spin  $\vec{S}$ , which, without loss of generality, we can take to be at the origin. The impurity spin is treated as point-like so that only the  $s$ -wave component of the conduction electrons interacts with it. By restricting to the  $s$ -wave channel, we effectively make the problem one dimensional. The radial dimension can be treated as running from  $r=0$  to  $r=\infty$ , with  $k$  species of nonchiral Dirac fermions  $\psi_i^{R,L}(r)$ ,  $i=1,2,\dots,k$  on the half line. They do not interact with each other but interact with a spin  $\vec{S}$  at  $r=0$ . Alternatively, we can formulate the model in terms of  $k$  species of chiral Dirac fermions on the line full line by allowing  $r$  (which we rename as  $x$ ) to run from  $-\infty$  to  $\infty$  and flipping the right-moving fermions onto the negative axis  $\psi_i^L(-x) \equiv \psi_i^R(x)$  for  $x>0$ . Thus, the  $k$ -channel Kondo Hamiltonian has  $k$  chiral Dirac fermions, interacting with the spin,

$$H_K = \sum_{i=1}^k \left\{ H_0[\psi_i] + \frac{\lambda_{\perp i}}{2} [J_i^+(0)S^- + \text{H.c.}] + \lambda_{zi} J_i^z(0)S^z \right\} + hS^z, \quad (17)$$

where  $H_0$  describes the  $k$  channels of the conduction electrons  $\psi_i$  (not parafermions)

$$H_0[\psi] = \int dx v_F \psi^\dagger i \partial_x \psi. \quad (18)$$

The local moment  $\vec{S}$  is situated at the origin. The local conduction-electron spin density in channel  $i$  is given by

$$J_i^-(x) = : \psi_{i\downarrow}^\dagger(x) \psi_{i\uparrow}(x) :,$$

$$J_i^+(x) = : \psi_{i\uparrow}^\dagger(x) \psi_{i\downarrow}(x) :,$$

$$J_i^z(x) = \frac{1}{2} : [\psi_{i\uparrow}^\dagger(x) \psi_{i\uparrow}(x) - \psi_{i\downarrow}^\dagger(x) \psi_{i\downarrow}(x)] :, \quad (19)$$

where  $::$  denotes normal ordering. The model is said to be *exchange isotropic* if  $\lambda_{\perp i} = \lambda_{zi}$ , and *channel symmetric* if  $\lambda_{\perp i} = \lambda_{\perp}$  and  $\lambda_{zi} = \lambda_z$  for all channels  $i$ . The last term in Eq. (17) is present if there is an externally applied magnetic field along the  $z$  direction.

The ground-state properties of Eq. (17) generally depend on  $k$ , the size of the spin  $\vec{S}$ ; the value of the external field  $h$ ; the presence/absence of anisotropy in the exchange and asymmetry in the channel coupling; and the signs of the products  $(\lambda_{\perp i})^2 \lambda_{zi}$ . Fortunately, we need only consider a small subset of the possible cases. From the point of view of ‘‘exotic’’ physics, one of the most important cases is when  $S=1/2$ ,  $h=0$ , and the model is *channel symmetric* (exchange anisotropy is unimportant, i.e., irrelevant<sup>81</sup> in the renormalization group sense). In this case, the ground state exhibits non-Fermi-liquid properties for  $k \geq 2$  in the channel symmetric case. However, these non-Fermi-liquid fixed points are unstable to channel asymmetry, which drives the system away from the channel symmetric limit (and to the  $k=1$  Fermi-liquid fixed point). We note that in our scenario discussed in Sec. IV and shown in Fig. 1, the ‘‘ $k$ ’’ appearing here is exactly the same  $k$  that appears in the RR states at filling fraction  $\nu = \frac{k}{k+2}$  or the  $\overline{\text{RR}}$  states at filling fraction  $\nu = \frac{2}{k+2}$ .

Furthermore, as we will see, our model automatically maps to the channel symmetric case.

### B. Emery-Kivelson solution of $H_K$ for $k=2$ and $S=1/2$

The Emery-Kivelson (EK) analysis<sup>82</sup> of the two-channel  $S=1/2$  Kondo model plays a central role in our work so we briefly review it here. EK solved the two-channel Kondo model by mapping it onto a Majorana resonant level model. (This gives a clue to the relation with our problem since Majorana fermions are an important part of the edge theories of the MR Pfaffian state and the anti-Pfaffian state.) They studied the properties of the resonant level model and then mapped them back onto the original Kondo model to determine quantities such as the impurity susceptibility and ground-state entropy. In Sec. IV, we will do the reverse: start with a particular resonant level model and ask ‘‘to what impurity (Kondo) model does it correspond?’’ We note, however, that not all resonant level models immediately map to a Kondo model, as we show explicitly in the case of a quantum dot coupled to *two* bulk fractional quantum Hall reservoirs.

The EK solution begins by bosonizing the electron operators,<sup>82</sup>

$$\psi_{i\sigma}^\dagger(x) = \frac{1}{\sqrt{2\pi a_0}} e^{i\phi_{i\sigma}(x)}, \quad (20)$$

where  $\sigma = \uparrow, \downarrow$ . In terms of these bosons, the local conduction-electron spin density is

$$J_i^\pm(x) = \frac{1}{2\pi a_0} e^{i\sqrt{2}\phi_{i-}(x)}, \quad J_i^z(x) = \frac{1}{\sqrt{2}\pi} \partial_x \phi_{i-}(x), \quad (21)$$

where

$$\phi_{i\pm}(x) = \frac{1}{\sqrt{2}} [\phi_{i\uparrow}(x) \pm \phi_{i\downarrow}(x)], \quad (22)$$

$a_0$  is a short-distance cutoff. We keep our normalization convention from before so that the electron operator has scaling dimension 1/2 and the currents have scaling dimension 1. Focusing on the two-channel case ( $i=1,2$ ), EK then introduce ‘‘spin’’ and ‘‘spin-flavor’’ bosons as the following linear combinations:

$$\phi_s(x) = \frac{1}{\sqrt{2}} [\phi_{1-}(x) + \phi_{2-}(x)], \quad (23)$$

$$\phi_{s'}(x) = \frac{1}{\sqrt{2}} [\phi_{1-}(x) - \phi_{2-}(x)], \quad (24)$$

which, as we will see momentarily, couple to the impurity spin while the fields

$$\phi_c(x) = \frac{1}{\sqrt{2}} [\phi_{1+}(x) + \phi_{2+}(x)], \quad (25)$$

$$\phi_f(x) = \frac{1}{\sqrt{2}}[\phi_{1+}(x) - \phi_{2+}(x)] \quad (26)$$

are not coupled to it. In terms of these fields, the Hamiltonian (17) takes the form,

$$\begin{aligned} H_K = & H_0[\phi_c] + H_0[\phi_f] + H_0[\phi_s] + H_0[\phi_{sf}] \\ & + \left( \frac{e^{i\phi_s(0)}}{2\pi a_0} \{ \lambda_{\perp+} \cos[\phi_{sf}(0)] + i\lambda_{\perp-} \sin[\phi_{sf}(0)] \} S^- \right. \\ & \left. + \text{H.c.} \right) + \frac{1}{\sqrt{\pi}} [\lambda_{z+} \partial_x \phi_s(0) + \lambda_{z-} \partial_x \phi_{sf}(0)] S^z + hS^z, \end{aligned} \quad (27)$$

where

$$\lambda_{\perp\pm} = \frac{1}{2}(\lambda_{\perp 1} \pm \lambda_{\perp 2}), \quad \lambda_{z\pm} = \frac{1}{2}(\lambda_{z1} \pm \lambda_{z2}). \quad (28)$$

Since  $\phi_c$  and  $\phi_f$  do not couple to the impurity spin, we ignore them from now on.

In order to map the Kondo Hamiltonian onto a resonant level model coupled to free fermions, EK perform a unitary transformation generated by

$$U = e^{iaS^z\phi_s(0)}, \quad (29)$$

in order to decouple  $\phi_s$  and thereby change the scaling dimension of the transverse exchange coupling to 1/2. Under the unitary transformation generated by Eq. (29),

$$\begin{aligned} H_K \rightarrow U^\dagger H_K U = & H_0[\phi_s] + H_0[\phi_{sf}] \\ & + \left( \frac{e^{i(1-\alpha)\phi_s(0)}}{2\pi a_0} \{ \lambda_{\perp+} \cos[\phi_{sf}(0)] + i\lambda_{\perp-} \sin[\phi_{sf}(0)] \} S^- \right. \\ & \left. + \text{H.c.} \right) + \frac{1}{\sqrt{\pi}} [\tilde{\lambda}_{z+} \partial_x \phi_s(0) + \lambda_{z-} \partial_x \phi_{sf}(0)] S^z + hS^z, \end{aligned} \quad (30)$$

where

$$\tilde{\lambda}_{z+} = \lambda_{z+} - \alpha v_F. \quad (31)$$

By choosing  $\alpha=1$ , one eliminates the coupling of the  $\phi_s$  field to  $S^\pm$  and sets the scaling dimension of the transverse exchange coupling to 1/2 so that the  $\phi_{sf}$  sector can be re-fermionized,

$$\psi_{sf}^\dagger(x) = e^{i\pi d^\dagger d} \frac{1}{\sqrt{2\pi a_0}} e^{i\phi_{sf}(x)}, \quad (32)$$

where a Klein factor  $e^{i\pi d^\dagger d}$  is inserted to ensure the proper anticommutation relations between the  $\psi_{sf}(x)$  fermions and the fermions introduced to describe the local magnetic moment,

$$S^+ = d^\dagger, \quad S^- = d, \quad S^z = d^\dagger d - 1/2. \quad (33)$$

The operators  $d, d^\dagger$  satisfy the anticommutation rules  $\{d, d^\dagger\} = 1$ . In this representation, the Hamiltonian takes the form

$$\begin{aligned} H_K = & H_0[\phi_s] + H_0[\psi_{sf}] + h(d^\dagger d - 1/2) + \frac{\lambda_{\perp+}}{\sqrt{8\pi a_0}} [\psi_{sf}^\dagger(0) \\ & + \psi_{sf}(0)](d - d^\dagger) + \frac{\lambda_{\perp-}}{\sqrt{8\pi a_0}} [\psi_{sf}^\dagger(0) - \psi_{sf}(0)](d + d^\dagger) \\ & + \left[ \tilde{\lambda}_{z+} \frac{\partial_x \phi_s(0)}{\sqrt{\pi}} + \lambda_{z-} \psi_{sf}^\dagger(0) \psi_{sf}(0) \right] (d^\dagger d - 1/2). \end{aligned} \quad (34)$$

At its generalized Toulouse point,  $\tilde{\lambda}_{z+} = \lambda_{z-} = 0$ , the Hamiltonian (34) is quadratic and, therefore, exactly solvable.<sup>82</sup>

Focusing for the moment on the Toulouse limit, the EK solution proceeds by introducing a Majorana representation for the local moment fermions and the spin-flavor fermions,

$$d = \frac{1}{\sqrt{2}}(a + ib), \quad (35)$$

$$\psi_{sf}(x) = \frac{1}{\sqrt{2}}[\zeta_1(x) + i\zeta_2(x)], \quad (36)$$

where  $a, b, \zeta_1, \zeta_2$  are Majorana fermions. In this representation, the Hamiltonian (apart from the degrees of freedom that decouple from the impurity spin) takes the form,

$$\begin{aligned} H_K = & H_0[\zeta_1] + H_0[\zeta_2] + h(iab - 1/2) + i \left( \frac{\lambda_{\perp+}}{\sqrt{2\pi a_0}} \right) \zeta_1(0)b \\ & + i \left( \frac{\lambda_{\perp-}}{\sqrt{2\pi a_0}} \right) \zeta_2(0)a, \end{aligned} \quad (37)$$

which describes two Majorana resonant levels ( $a$  and  $b$ ) coupled to *two* different Majorana modes ( $\zeta_1$  and  $\zeta_2$ ) originating from the Fermi sea. Note that the magnetic field couples the two Majorana resonant levels even at the Toulouse point.

Let us now focus on Eq. (37) with  $h=0$ . In this case, the Majorana modes completely decouple from each other, and the Kondo Hamiltonian reduces to a sum of two Hamiltonians each describing the coupling of a different Majorana resonant level to a different Majorana bath. It can be shown that the one-channel Kondo model at its Toulouse point can be mapped into just such a form.<sup>82,83</sup> (The transverse coupling in the one-channel model takes the form  $\psi^\dagger d + d^\dagger \psi$  which is equivalent to Eq. (37) with a constant shift of  $\phi_{sf}$ ) Therefore, if  $\lambda_{\perp+} \neq 0$  and  $\lambda_{\perp-} \neq 0$ , the two-channel Kondo model exhibits one-channel Kondo behavior.<sup>83,84</sup> On the other hand, if the model is *channel* symmetric ( $\lambda_{\perp-} = 0$  and  $\lambda_{z-} = 0$ ) then only the “ $b$ ” Majorana fermion couples to the conduction electrons and non-Fermi-liquid properties result. For example, there is a nontrivial ground-state entropy of

$$S_{\text{imp}}(0) = \frac{1}{2} \ln(2) \quad (38)$$

and an impurity susceptibility,

$$\chi_{\text{imp}}(T) \propto \ln(T_K/T), \quad (39)$$

where  $T_K$  is the Kondo temperature.<sup>82,85</sup>



Detailed analysis<sup>81–85</sup> of the full Hamiltonian (34) leads to the following key results: the two-channel non-Fermi-liquid fixed point at  $\tilde{\lambda}_{z+}=\lambda_{z-}=\lambda_{\perp-}=h=0$  is (i) stable to *exchange* anisotropy,  $\lambda_{\perp+}\neq\lambda_{z+}\neq 0$ , (ii) unstable to *channel* anisotropy,  $\lambda_{z-}\neq 0$  or  $\lambda_{\perp-}\neq 0$ , and (iii) unstable to a finite magnetic field  $h\neq 0$ . In case (ii), the low-energy properties are controlled by the one-channel Kondo-model fixed point (provided the effective coupling is antiferromagnetic) at which the ground-state entropy is zero and the impurity susceptibility is a constant. However, provided the asymmetry is not too large, the two-channel behavior may survive over a fairly large energy scale before crossing over to the one-channel behavior at the lowest energies.<sup>83,84</sup> For case (iii), the low-energy behavior is described by a potential scattering problem with different phase shifts for up- and down-spin electrons.<sup>81</sup>

### C. Affleck-Ludwig solution of $H_K$ for general $k$ and $S=1/2$

For  $k>2$ -channel Kondo models, the Abelian bosonization method of EK is not as helpful (except for the  $k=4$ -channel case<sup>83</sup>) so other approaches are needed. For our later discussion, the boundary CFT solution to the  $k$ -channel (including single-channel) Kondo model pioneered by Affleck and Ludwig<sup>86–88</sup> turns out to be particularly useful. Since the RR and  $\overline{RR}$  boundary excitations are described by a CFT (the same CFT, in fact, that appears in the multichannel Kondo model), this approach provides a natural link between multichannel Kondo models and non-Abelian quantum Hall states. The shared CFT is the “deep” reason we can establish a mapping between the resonant level model at the edge of a RR or  $\overline{RR}$  state and the multichannel Kondo model. We emphasize that the boundary CFT method covers all cases, even the ( $k=1$ )-channel Kondo model.

The boundary CFT solution to Eq. (17) begins with a conformal decomposition of the conduction electrons.<sup>86–88</sup> The conduction-electron term represented by  $H_0$  is expressed as a sum of three terms (referred to as a “conformal embedding”) that describe “charge,” spin, and “flavor” sectors. These sectors possess current operators with U(1), SU(2), and SU( $k$ ) symmetry, respectively. Specifically, one has

$$\sum_{i=1}^k H_0[\psi_i] = H[U(1)] + H[SU(2)_k] + H[SU(k)_2]. \quad (40)$$

The identity (40) is typically motivated<sup>89</sup> by noting that the central charges on the left- and right-hand sides are equal:  $2k=1+\frac{3k}{k+2}+\frac{2(k^2-1)}{k+2}$ . Conformal embedding is useful for the  $k$ -channel Kondo model because it is only the SU(2) <sub>$k$</sub>  currents that couple to the local moment  $\vec{S}$ : the conduction-electron terms with U(1) and SU( $k$ ) symmetry thus play no role in the impurity physics. (Recall that the  $Z_k$  parafermions in the RR and  $\overline{RR}$  states are constructed from the coset SU(2) <sub>$k$</sub> /U(1). Thus, SU(2) <sub>$k$</sub>  plays a key role in both the non-Abelian fractional quantum Hall states of interest here and the  $k$ -channel Kondo model.)

In terms of the  $k$  channels of the conduction electrons, the SU(2) <sub>$k$</sub>  spin currents can be expressed as

$$\vec{J}(x) = \sum_{i=1}^k : \psi_{\alpha,i}^\dagger(x) \frac{\vec{\sigma}_{\alpha\beta}}{2} \psi_{\beta,i}(x) :, \quad (41)$$

where  $\vec{\sigma}$  is the vector of Pauli spin matrices. The Fourier components of the SU(2) <sub>$k$</sub>  spin currents satisfy the Kac-Moody algebra,

$$[J_n^a, J_m^b] = i \epsilon^{abc} J_{n+m}^c + \frac{nk}{2} \delta^{ab} \delta_{n+m,0}. \quad (42)$$

Thus, the currents can also be expressed in terms of a parafermion and a boson, as in Eq. (9).

The next step is to study the coupling of the magnetic impurity to the spin currents. Let us first assume the coupling of the spin currents to the local moment  $\vec{S}$  is channel symmetric. The Kondo Hamiltonian is then (neglecting parts that decouple from the spin),

$$H_K = \frac{2\pi v_F}{k+2} \int_0^L dx : \vec{J}(x) \cdot \vec{J}(x) : + \lambda_{\perp} [J^+(0)S^- + J^-(0)S^+] + \lambda_z J^z(0)S^z + hS^z. \quad (43)$$

To illustrate the key features of the Affleck-Ludwig boundary CFT solution it is useful to specialize to the exchange isotropic limit in zero magnetic field.<sup>86–88</sup> The relevance/irrelevance of exchange anisotropy and channel asymmetry, as well as an externally applied magnetic field, can be analyzed about this point.<sup>81</sup> Going to the Sugawara form of the Hamiltonian in Fourier space, we have

$$H_K = \frac{2\pi v_F}{L} \left( \sum_{n=-\infty}^{\infty} \frac{1}{k+2} : \vec{J}_{-n} \vec{J}_n : + \lambda_K \sum_{n=-\infty}^{\infty} \vec{J}_n \cdot \vec{S} \right), \quad (44)$$

where  $\lambda_K$  is the effective Kondo coupling. The crucial point is that when  $\lambda_K=2/(k+2)$  (which corresponds to the low-temperature fixed point), the Hamiltonian (44) becomes equivalent, after completing the square, to an uncoupled spin but with shifted current operators,

$$\vec{J}_n = \vec{J}_n + \vec{S}. \quad (45)$$

In other words, at the low-energy fixed point, the local moment is “absorbed” into the spin currents. This is referred to as the “fusion hypothesis.” In boundary CFT terms, it can be stated in the following way. Let us fold the system again, so that we have nonchiral fields on the half plane  $x>0$ ,  $-\infty < \tau < \infty$ . Each possible conformally invariant boundary condition at the boundary of the half plane,  $x=0$ , corresponds to a primary field of the theory (this is true in a large class of theories, namely, those with diagonal partition functions, which includes the models discussed here). When the interaction is tuned to zero,  $\lambda_K=0$ , the boundary condition corresponds to the identity operator in SU(2) <sub>$k$</sub> . According to the fusion hypothesis, the boundary condition at the infrared fixed point is obtained by fusing the identity with the spin-1/2 primary field (or, more generally, the spin- $S$  primary field), i.e., the boundary condition is associated with the spin-1/2 primary field. This leads to an entropy gain of  $\ln\{2 \cos[\pi/(k+2)]\}$  associated with the conformal boundary condition. However, since the impurity spin, with its  $\ln 2$

entropy is screened, the net entropy change is  $\Delta S = -\ln 2 + \ln\{2 \cos[\pi/(k+2)]\}$ . If we separate bulk and impurity so that the initial impurity entropy is  $\ln 2$  (how we choose to divide the total entropy of the system into bulk and impurity is, in part, a convention), then the entropy in the infrared is

$$S_{\text{imp}}(0) = \ln\{2 \cos[\pi/(k+2)]\}. \quad (46)$$

We have already seen this scenario in different language in Emery and Kivelson's solution of the  $k=2$  case. Let us consider the channel symmetric case as the generalized Toulouse point so that  $\lambda_{\perp+}=0$ . For  $\lambda_{\perp+}=0$ , the Majorana fermion  $\zeta_1$  in Eq. (37) has boundary condition  $\zeta_{1R}(0)=\zeta_{1L}(0)$ , which corresponds to fixed boundary condition in the Ising model. However, for  $\lambda_{\perp+}>0$ , which flows to the infrared fixed point  $\lambda_{\perp+}=\infty$ , the boundary condition is  $\zeta_{1R}(0)=-\zeta_{1L}(0)$  [as may be seen by direct solution of the quadratic Hamiltonian (37)]. This corresponds to free boundary condition in the Ising model, which has a boundary entropy which is higher by  $\ln\sqrt{2}$ . However, since the impurity spin's entropy  $\ln 2$  is lost, there is a net entropy loss of  $\ln\sqrt{2}$ .

As a result of the change in boundary condition in the  $SU(2)_k$  sector while the  $U(1)$  and  $SU(k)_2$  sectors are unaffected, correlation functions of electron operators (which combine these three sectors) become nontrivial. For instance, the impurity susceptibility for the  $k$ -channel Kondo model is

$$\chi_{\text{imp}}(T) \propto T^{-(k-2)/(k+2)}. \quad (47)$$

The fusion hypothesis has been tested against exact Bethe-ansatz results<sup>90</sup> and numerical renormalization group studies and is now believed to be on quite solid ground.<sup>81</sup>

The conclusions reached with the boundary CFT analysis<sup>81</sup> for  $S=1/2$  and general  $k \geq 2$  are the same as those reached by the EK analysis for  $k=2$  as far as symmetry-breaking perturbations are concerned. The  $k$ -channel non-Fermi-liquid fixed point is (i) stable to *exchange* anisotropy, (ii) unstable to *channel* asymmetry, and (iii) unstable to a finite magnetic field. In case (ii), the low-energy properties are controlled by the one-channel Kondo-model fixed point (provided the effective coupling is antiferromagnetic). For case (iii), the low-energy behavior is described by a potential scattering problem with different phase shifts for up and down electrons.<sup>81</sup>

#### IV. EXOTIC RESONANT LEVEL MODELS

Having laid the necessary ground work for our study in Secs. II and III, we are now ready to investigate the physics of a quantum dot (modeled by a single resonant level) coupled by electron tunneling to a Read-Rezayi quantum Hall state or its particle-hole conjugate. As we mentioned in Sec. I, our main result is that the physics is governed by some version of the multichannel Kondo model.

Specifically, we study the setup shown in Fig. 1. The bulk quantum Hall state on the left is assumed to be in a non-Abelian fractional quantum Hall state at  $\nu=2+k/(k+2)$  or  $\nu=2+2/(k+2)$ . A quantum point contact may be used to pinch off a finite region of the quantum Hall fluid and form a quantum dot separated from the bulk by a tunneling barrier. We assume that the lower two Landau levels are not pinched

off and therefore do not backscatter at the point contact; the barrier region is assumed to have  $\nu=2$ , effectively forming a vacuum for the RR or  $\overline{\text{RR}}$  state. For a finite system (the quantum dot), the edge modes acquire a discrete spectrum.<sup>91,92</sup> We focus on fluctuations of the quantum dot charge  $Q \equiv e\langle \hat{N}_e \rangle$  near a degeneracy point in the energy:  $E(N_e, S, B) = E(N_e + 1, S, B)$ , where  $N_e$  is the number of electrons on the quantum dot. The energy  $E$  depends on dot area  $S$ , which may be altered by a gate potential shown in Fig. 1, and on the magnetic field  $B$ . Adjusting either  $S$  or  $B$  may be used to achieve the desired degeneracy and also to slightly tune away from it. The charge of the dot can be measured capacitively.<sup>40,41,93,94</sup>

We assume energy and temperature scales are much less than the level spacing of the dot edge states. Then only two levels on the quantum dot (the degenerate or nearly degenerate ones) are important. These two levels act as an effective, local spin-1/2 degree of freedom. The mapping of charge occupation on a quantum dot to an effective spin degree of freedom is not new<sup>95-101</sup> and a number of related studies of quantum dots coupled to interacting one-dimensional systems have been carried out.<sup>102-107</sup> However, the physics resulting from coupling such a quantum dot to a RR or  $\overline{\text{RR}}$  state has only recently been touched upon in earlier work by the present authors.<sup>35</sup>

For the remainder of this section we will study the Hamiltonian,

$$H = H_{\text{edge}} + H_{\text{dot}} + H_{\text{tun}}, \quad (48)$$

where  $H_{\text{edge}}$  is the edge Hamiltonian of a RR or  $\overline{\text{RR}}$  state (including the  $k=2$  MR and  $\overline{\text{MR}}$  states). Here and henceforth, we ignore the two filled lower Landau levels. This is justified by the sequence of modes pinched off in a point contact.<sup>11,108</sup> The dot Hamiltonian is

$$H_{\text{dot}} = \epsilon_d d^\dagger d, \quad (49)$$

where  $d(d^\dagger)$  is the fermionic annihilation (creation) operator for this state. The level energy  $\epsilon_d=0$  at the degeneracy point and  $\epsilon_d \neq 0$  when one is tuned away from degeneracy. [Since we are focused on a level degeneracy point, the standard Coulomb charging term of the form  $\frac{E_C}{2}(\hat{N}-N_e)^2$  does not play a key role in the physics of interest to us, although such a term is implicit in our analysis.] The final term in Eq. (48),  $H_{\text{tun}}$ , describes the tunneling of *electrons* between the edge of the bulk quantum Hall state and the quantum dot. Its specific form depends on the quantum Hall state of the bulk, and it will determine the properties of the emergent Kondo model. We will consider a number of important cases below.

##### A. Tunneling from the MR Pfaffian state

The edge theory for the Pfaffian state is the sum of a free, charged chiral bosonic sector, and a neutral Majorana (the " $\mathbb{Z}_2$  parafermion") sector. The edge Hamiltonian takes the form

$$H_{\text{MR edge}} = \int dx \left\{ v_c \frac{2}{4\pi} [\partial_x \phi(x)]^2 + i v_n \psi \partial_x \psi \right\}, \quad (50)$$

where  $v_n < v_c$  generally holds,  $\phi$  is a real chiral boson, and  $\psi$  is a chiral Majorana fermion. The Hamiltonian (50) is equivalent to the sum of Eqs. (5) and (6). The tunneling Hamiltonian for the MR state takes the form

$$H_{\text{tun}} = t [d^\dagger \Psi_{e,\text{MR}}(0) + \Psi_{e,\text{MR}}^\dagger(0) d] + V d^\dagger d \Psi_{e,\text{MR}}^\dagger(0) \Psi_{e,\text{MR}}(0), \quad (51)$$

where  $t$  is the tunneling amplitude to the dot (which we have, without loss of generality, assumed to be real). The tunneling is assumed to occur at the origin,  $x=0$ . The parameter  $V$  represents the strength of the Coulomb repulsion between the edge and the dot, and electron operator is given by Eq. (7),  $\Psi_{e,\text{MR}}^\dagger = \psi e^{i2\phi}$ . It has scaling dimension  $3/2$ .

As a result of the scaling dim of  $\Psi_{e,\text{MR}}$ ,  $t$  is naively irrelevant for small  $V$ ,

$$\frac{dt}{d\ell} = -\frac{1}{2}t + \frac{tV}{\pi v_c} + \mathcal{O}(t^3). \quad (52)$$

However, for  $V$  sufficiently large,  $t$  flows to the two-channel Kondo fixed point, not to  $t=0$ . To see this, we apply to  $H$  a unitary transformation similar to that of Emery and Kivelson,

$$U = e^{2id^\dagger d\phi(0)}. \quad (53)$$

This rotates  $\phi(0)$  out of the tunneling term.  $H$  now takes the form

$$UHU^\dagger = H_{\text{MR edge}} + H_{\text{dot}} + t\psi(d - d^\dagger) + (V - 2v_c)d^\dagger d \partial_x \phi(0). \quad (54)$$

For  $V - 2v_c = 0$ , this is a purely quadratic theory which can be solved exactly. Thus,  $t$  is clearly relevant in this limit; it is actually relevant over a range of values of  $V$ , as we discuss below. Note that only the Majorana combination  $d - d^\dagger$  couples to the quantum Hall edge. This is precisely the same feature which leads to non-Fermi-liquid behavior in the two-channel Kondo problem,<sup>82</sup> as can be seen by direct comparison with Eq. (37) for zero field,  $h=0$ , and channel *isotropy*,  $\lambda_{\perp} = 0$ .

Equation (54) therefore establishes that the tunneling of electrons from a MR state to a quantum dot is described by the channel *symmetric* two-channel Kondo model. Having the EK solution in hand, it is evident that the channel isotropy in Eq. (54) follows from the unique Majorana fermion on the edge of the MR state,  $\psi$ , that appears in the electron operator. The absence of another edge Majorana to couple to the “ $d + d^\dagger$ ” combination follows directly from the topological properties of the MR Pfaffian state. Therefore, the channel symmetric nature of the emergent two-channel Kondo model is *topologically protected* by the MR state. Typically, the channel symmetric limit requires delicate fine tuning and is very difficult to realize in experiment<sup>109</sup> but the topological protection of our situation is a boon. Most importantly, though, the non-Fermi-liquid physics of the emergent two-channel Kondo model leads, in principle, to a way to *identify* the fractional quantum Hall state as the MR Pfaffian state.

Another way to see the connection to the two-channel Kondo model is to reverse the mapping of EK. To this end, it is useful to represent the two-level system on the dot by a spin:  $S^\dagger = \eta d^\dagger$ ,  $S^- = d \eta$ , and  $S^z = d^\dagger d - 1/2$ , where  $\eta$  are Klein factors that ensure the proper commutation relations are achieved.<sup>101</sup> The  $\eta$  have the property that  $\eta^\dagger = \eta$ ,  $\eta^2 = 1$  and they anticommute with fermions, i.e., their properties are like nondynamical Majorana fermions. We apply a unitary transformation  $U = e^{i\alpha S^z \phi(0)}$  to  $H$  as before, but take  $\alpha = 2 - \sqrt{2}$ , to partially rotate  $\phi(0)$  in the tunneling term, giving

$$UHU^\dagger = H_{\text{MR edge}} + \epsilon_d S^z + (V - v_c \alpha) S^z \partial_x \phi(0) + t(\eta \psi e^{-i\sqrt{2}\phi(0)} S^\dagger + \psi \eta e^{i\sqrt{2}\phi(0)} S^-). \quad (55)$$

When the term proportional to  $t$  in Eq. (55) is computed to any order in perturbation theory, only even powers of  $S^- S^\dagger$  and  $S^\dagger S^-$  will appear and the Klein factor  $\eta$  will disappear due to the relation  $\eta^2 = 1$ . We may therefore safely drop it altogether, having already served its role in transforming fermionic operators  $d$  to spin operators  $S$ ,

$$UHU^\dagger = H_{\text{MR edge}} + \epsilon_d S^z + (V - v_c \alpha) S^z \partial_x \phi(0) + t(\psi e^{-i\sqrt{2}\phi(0)} S^\dagger + \psi e^{i\sqrt{2}\phi(0)} S^-). \quad (56)$$

We now compare this to the channel symmetric Kondo model,

$$H_{\text{imp}} = \lambda_{\perp} [J^+(0) S^- + J^-(0) S^+] + \lambda_z J^z(0) S^z + h S^z, \quad (57)$$

where  $\vec{S}$  is the impurity spin;  $\vec{J}(0)$  is conduction-electron spin density at the impurity site;  $\lambda_{\perp}$  and  $\lambda_z$  are the exchange couplings, which are not assumed to be equal; and  $h$  is the magnetic field. The currents  $J^a$  can be expressed in terms of a Majorana fermion,  $\psi$ , and a free boson  $\phi$ ,

$$J^+ = \sqrt{2} \psi e^{i\sqrt{2}\phi}, \quad J^- = \sqrt{2} \psi e^{-i\sqrt{2}\phi}, \quad J^z = \sqrt{2} \partial_x \phi, \quad (58)$$

which is a special case of Eq. (9). The factors of  $\sqrt{2}$  are present in this expression because the boson  $\phi$  is normalized here according to Eq. (50) so that  $\text{dim}[e^{i\phi}] = 1/4$ , rather than  $\text{dim}[e^{i\phi}] = 1/k = 1/2$ , as assumed in Eq. (9).

Substituting Eq. (58) into Eq. (57), we see that our problem (56) maps onto the two-channel Kondo model with anisotropic exchange if we identify  $\lambda_{\perp} = t$ ,  $\sqrt{2}\lambda_z = V - (2 - \sqrt{2})v_c$ , and  $h = \epsilon_d$ . For  $\lambda_z < 0$ , the Kondo model is ferromagnetic. In the ferromagnetic Kondo model, the coupling to the impurity is irrelevant, as we naively expected above [Eq. (52)]. However, when  $V$  is sufficiently large,  $\lambda_z > 0$ , corresponding to the antiferromagnetic Kondo model. In Sec. V, we discuss the regime  $V - (2 - \sqrt{2})v_c > 0$  in terms of realistic parameters for experiments. In this case, the Hamiltonian is controlled in the infrared by the exchange and channel symmetric antiferromagnetic spin-1/2 two-channel Kondo fixed point.<sup>81</sup> The non-Fermi-liquid behaviors of the spin susceptibility and magnetic field dependence of the zero-temperature magnetization translate to the charge susceptibility and charge of the quantum dot,<sup>87</sup>

$$\chi_{\text{charge}} \propto \ln(T_K/T),$$

$$\Delta Q \propto V_G \ln(k_B T_K / e^* V_G). \quad (59)$$

Here,  $\Delta Q = Q - e(N_e + \frac{1}{2})$  is the charge on the dot measured relative to the average electron number at the degeneracy point of the energy. The Kondo temperature  $T_K$  is the energy scale at which the coupling becomes  $\mathcal{O}(1)$ . In the exchange isotropic Kondo problem, in which the interaction  $\lambda$  is marginal at tree level,  $T_K \propto \exp(-\pi v_n / \lambda)$ . In a Majorana fermion resonant level model, in which the tunneling  $\lambda$  has scaling dimension  $1/2$ ,  $T_K \propto \lambda^2$ . In our problem, however, due to the renormalization group (RG) Eq. (52), together with  $dV/d\ell = t^2 / \pi v_c$ , we have a Kosterlitz-Thouless (KT)-type set of RG equations. For  $V \gg \pi v_c / 2$ , the flows in the  $V$ - $t$  plane are nearly vertical so that  $T_K \propto t^{1/[(V/\pi v_c) - (1/2)]}$ . However, closer to the Kosterlitz-Thouless point, we expect the flow to strong coupling to be slower so that, for instance, along the KT separatrix, an exponential form such as  $T_K \propto e^{-c/t}$  would hold for some constant  $c$ .

### B. Tunneling from the RR state

The analysis for electron tunneling between a RR state for general  $k$  and the quantum dot is almost identical to that for  $k=2$ . The edge theory for filling fraction  $\nu = k/(k+2)$  is

$$H_{\text{edge}} = \frac{(k+2)/k}{4\pi} v_c \int dx [\partial_x \phi(x)]^2 + H_{Z_k}, \quad (60)$$

where  $H_{Z_k}$  describes the (interacting, except for  $k=2$ ) neutral  $Z_k$  parafermionic sector of the edge. The tunneling term takes the form

$$H_{\text{tun}} = t [d^\dagger \Psi_{e,\text{RR}}(0) + \Psi_{e,\text{RR}}^\dagger(0) d] + V d^\dagger d \Psi_{e,\text{RR}}^\dagger(0) \Psi_{e,\text{RR}}(0), \quad (61)$$

and  $H_{\text{dot}}$  is the same as before. The electron operator  $\Psi_e^\dagger = \psi_1 e^{i(k+2/k)\phi}$ , as given in Eq. (10), and has scaling dimension  $3/2$ . The crucial difference between the RR state and the MR state is that the  $Z_k$  parafermion  $\psi_1$  has replaced the  $Z_2$  Majorana fermion.

As before, we apply a unitary transformation  $U = e^{i\alpha S^z \phi(0)}$  to  $H$ , which now takes the form,

$$U H U^\dagger = H_{\text{edge}} + H_{\text{dot}} + (V - v_c \alpha) S^z \partial_x \phi(0) + t (\psi_1^\dagger e^{-i\tilde{\alpha}\phi(0)} S^+ + \psi_1 e^{i\tilde{\alpha}\phi(0)} S^-), \quad (62)$$

where  $\tilde{\alpha} \equiv \frac{k+2}{k} - \alpha$ . The choice  $\alpha^* = \frac{k+2}{k} - \sqrt{\frac{k+2}{k}} \sqrt{\frac{2}{k}}$  makes the connection to the  $k$ -channel Kondo problem explicit because the  $SU(2)_k$  current operators can be represented in terms of the  $Z_k$  parafermions,

$$J^+ = \sqrt{k} \psi_1 e^{i\beta\phi}, \quad J^- = \sqrt{k} \psi_1^\dagger e^{-i\beta\phi}, \quad J^z = \frac{k}{2} \beta \partial_x \phi, \quad (63)$$

where  $\beta = \sqrt{2(k+2)}/k$ . [As in the MR case, the boson  $\phi$  is normalized differently here than in Eq. (9): according to Eq. (60),  $\dim[e^{i\phi}] = k/2(k+2)$  rather than  $\dim[e^{i\phi}] = 1/k = 1/2$ , as assumed in Eq. (9).] Substituting these expressions into Eq. (57), we see that our problem (62) is equivalent to the channel symmetric  $k$ -channel Kondo problem if we identify  $\lambda_\perp = t$ ,  $\beta \lambda_z = V - v_c \alpha^*$ , and  $h = \epsilon_d$ . For  $V > v_c \alpha^*$ , this is the anti-

ferromagnetic Kondo problem, which has an intermediate coupling fixed point. Thus, we see that the Read-Rezayi states offer a scenario to realize the non-Fermi-liquid behavior of the  $k$ -channel Kondo model,

$$\chi_{\text{charge}} \propto T^{-(k-2)/(k+2)},$$

$$\Delta Q \propto V_G^{2/k}, \quad (64)$$

which would otherwise require an incredible amount of fine tuning for  $k \geq 3$ . Moreover, observing the predicted non-Fermi-liquid behavior would provide strong evidence that the quantum Hall state is of the Read-Rezayi type. Again, we emphasize that the non-Fermi-liquid physics of the channel symmetric  $k$ -channel Kondo model is *topologically protected* by the RR state. Its edge theory has a unique parafermion in the electron operator.

As we will now see, the non-Fermi-liquid physics is no longer topologically protected when one considers the particle-hole conjugate states. The basic reason is that the particle-hole conjugate states have multiple electron operators which need not have the same tunneling amplitudes.

### C. Tunneling from the MR anti-Pfaffian state

The edge theory of the MR anti-Pfaffian state is<sup>24,25</sup>

$$\mathcal{L}_{\overline{\text{MR}}} = \frac{2}{4\pi} \partial_x \phi (i\partial_\tau + v_c \partial_x) \phi + \sum_{a=1}^3 i \psi_a (-i\partial_\tau + v_n \partial_x) \psi_a, \quad (65)$$

as discussed in Sec. II. There are three different dimension-3/2 electron operators,  $\psi_a e^{2i\phi}$  for  $a=1,2,3$ . The tunneling Hamiltonian is

$$H_{\text{tun}} = \sum_{a=1}^3 (t_a \psi_a e^{-2i\phi} d^\dagger + \text{H.c.}) + V d^\dagger d \partial_x \phi. \quad (66)$$

Although the edge theory has an emergent  $SU(2)$  symmetry, and the three electron operators  $\psi_a$  form a triplet under this symmetry, this symmetry is not fundamental and the tunneling operators do not have to respect it.

Performing a unitary transformation, as before, to rotate out the  $\phi$  dependence of the first term, we obtain  $U H U^\dagger = H_{\text{edge}} + H_{\text{dot}} + \tilde{H}_{\text{tun}}$ , where

$$\tilde{H}_{\text{tun}} = \sum_{a=1}^3 (t_a \psi_a d^\dagger + \text{H.c.}) + (V - 2v_c) d^\dagger d \partial_x \phi$$

$$= i\chi_1 [\lambda_1 (d^\dagger - d)/i + \lambda_1' (d^\dagger + d)] + i\lambda_2 \chi_2 (d^\dagger + d) + (V - 2v_c) d^\dagger d \partial_x \phi, \quad (67)$$

where  $\chi_1 = u_a \psi_a / \sqrt{u^2}$ ,  $\chi_2 = w_a \psi_a / \sqrt{w^2}$ ,  $u_a = \text{Re } t_a$ ,  $v_a = \text{Im } t_a$ ,  $w_a = v_a - u_a (u \cdot v / u^2)$ ,  $\lambda_1 = \sqrt{u^2}$ ,  $\lambda_2 = \sqrt{w^2}$ ,  $\lambda_1' = u \cdot v / \sqrt{u^2}$ , and  $\{\chi_1, \chi_2\} = 0$ . Note that, for generic  $t_a$ , both  $a = (d^\dagger + d)/\sqrt{2}$  and  $b = (d^\dagger - d)/i\sqrt{2}$  couple to the edge modes, as in the *one-channel* Kondo model. Thus, we expect that Eq. (67) is also controlled by the one-channel Kondo fixed point.

This is in contrast to the Pfaffian case, in which only  $b$  couples to the edge modes, as in the channel symmetric two-

channel Kondo model. One might naively expect that both  $a$  and  $b$  would couple to the edge modes even in the Pfaffian case if the tunneling amplitude is  $t$  is not purely real. However, for  $t=|t|e^{i\theta}$  we could always find a linear combination  $a \cos \theta + b \sin \theta$  which does not couple. The problem in the anti-Pfaffian case is that the linear combination  $a \cos \theta_1 + b \sin \theta_1$  which does not couple to  $\psi_1$  will, in general, couple to  $\psi_2$  and  $\psi_3$ . Only in the special case in which all three tunneling amplitudes have the same phase will two-channel Kondo behavior result.

The charge susceptibility and charge of the quantum dot have the temperature and voltage dependence characteristic of a Fermi liquid,

$$\chi_{\text{charge}} \propto \text{const.}, \quad \Delta Q \propto V_G. \quad (68)$$

Consequently, measurements of the behavior of the dot would distinguish the Pfaffian and anti-Pfaffian states.

#### D. Tunneling from the $\overline{\text{RR}}$ state

The edge theory of the  $\overline{\text{RR}}$  state is given by the sum of Eq. (1) with  $\nu=2/(k+2)$  and Eq. (15). The tunneling Hamiltonian takes the form

$$H_{\text{tun}} = \sum_{m=-k/2}^{k/2} (t_m \chi_{j=k/2}^m e^{i(k+2/2)\phi} d + \text{H.c.}) + V d^\dagger d \partial_x \phi. \quad (69)$$

It is useful to rewrite this expression using the parafermion representation of  $\text{SU}(2)_k$  [e.g., using Eq. (9)]. Then  $\chi_{j=k/2}^m = \psi_{m+k/2} e^{-im\phi_\sigma/2}$ , where  $\psi_0 = \psi_k \equiv 1$  and  $\psi_1, \psi_2, \dots, \psi_{k-1}$  are the parafermion operators. The tunneling Hamiltonian then takes the form

$$H_{\text{tun}} = \sum_{m=-k/2}^{k/2} (t_m \psi_{m+k/2} e^{-im\phi_\sigma/2} e^{i(k+2/2)\phi} d + \text{H.c.}) + V d^\dagger d \partial_x \phi. \quad (70)$$

Separating the  $m=1-k/2$  term from the sum, we have

$$H_{\text{tun}} = t_{1-k/2} \psi_1 e^{i(k-2)\phi_\sigma/2} e^{i(k+2/2)\phi} d + \text{H.c.} \\ + V d^\dagger d \partial_x \phi \sum_{m=-k/2}^{k/2} ' (t_m \psi_{m+k/2} e^{-im\phi_\sigma/2} e^{i(k+2/2)\phi} d + \text{H.c.}), \quad (71)$$

where the prime on the summation indicates that the sum does not include the  $m=1-k/2$  term. Performing a unitary transformation, we can rewrite this as

$$H_{\text{tun}} = t_{1-k/2} \psi_1 d + \text{H.c.} + [V - (1+k/2)v_c] d^\dagger d \partial_x \phi + (1 \\ - k/2)v_n d^\dagger d \partial_x \phi_\sigma \\ + \sum_{m=-k/2}^{k/2} ' [t_m \psi_{m+k/2} e^{-i(m-1+k/2)\phi_\sigma/2} e^{i(k+2/2)\phi} d + \text{H.c.}]. \quad (72)$$

The first line of Eq. (72) is essentially the channel symmetric  $k$ -channel Kondo model, after unitary transformation, as in

Eq. (62) with  $\tilde{\alpha} = \frac{k+2}{2}$ . The second and third lines may be viewed as perturbations of this model. There are  $k$  couplings  $t_m$ ,  $m \neq 1-k/2$ , and it is tempting to identify them with the possible channel anisotropies in the  $k$ -channel Kondo problem. However, we do not have a direct mapping and, indeed, we do not expect a simple mapping of this form since channel asymmetry in the Kondo model would necessarily involve operators in the  $\text{SU}(k)_2$  flavor sector, which does not enter our model. However, the third line of Eq. (72) is certainly a relevant perturbation so it will drive the system away from the channel symmetric  $k$ -channel fixed point. It seems likely that it will drive the system to the one-channel fixed point, i.e., that there will be no boundary entropy left since that is the most stable fixed point and the  $k$  relevant operators in Eq. (72) would be expected to destabilize any other possible fixed point. However, whether or not the one-channel Kondo behavior governs the ultimate low-energy fixed point, tunneling to the RR state will not be governed by the channel symmetric  $k$ -channel Kondo model.

#### E. Other candidate states at $\nu=5/2$ and $\nu=12/5$

Before leaving the problem of a quantum dot coupled to a *single* edge of a fractional quantum Hall state, we briefly return to filling fraction  $\nu=5/2$ . Besides the Moore-Read Pfaffian and the anti-Pfaffian, there are at least three other candidate states for  $\nu=5/2$ , the non-Abelian  $\text{SU}(2)_2$  NAF state,<sup>36,37</sup> the Abelian  $K=8$  strong-coupling state,<sup>38</sup> and the Abelian Halperin (3,3,1) state.<sup>38</sup> While numerically, the Pfaffian and anti-Pfaffian appear to be favored,<sup>5-10</sup> it is useful to discuss what type of behavior we would expect in our setup for these other states. In a recent work by Bishara *et al.*, the signatures in quantum Hall interferometry for each of these five candidate states were discussed.<sup>27</sup>

In order to anticipate what behavior to expect for these other candidate states, it is useful to recall the feature that led to the stable two-channel Kondo behavior for the Pfaffian: a unique edge electron operator that was built in part from the Majorana fermion. Since the electron operator and the  $\text{SU}(2)$  currents only differ by the scaling dimension of the bosonic charge sector portion of the electron operator (which can be changed with a unitary transformation in the tunneling Hamiltonian as we saw earlier), the Pfaffian is guaranteed to map onto the channel isotropic version of the two-channel Kondo model. When the electron operator was no longer unique and/or was not built from a parafermion (as is the case for the MR or RR states) then the system generically flows to a single-channel Kondo model. We expect the single-channel Kondo fixed point to be the ultimate fate of the non-Abelian  $\text{SU}(2)_2$  NAF state, the Abelian  $K=8$  strong-coupling state, and the Abelian Halperin (3,3,1) state. This is true, even though in states like the (3,3,1) state where the spin degree of freedom is active and one would expect a two-channel Kondo model to be realized (because the situation is similar to that discussed in Refs. 96 and 97). However, this two-channel Kondo model will not be “topologically protected” in the way that it is for the Pfaffian. Because of residual Zeeman coupling to the electron’s spin in the quantum Hall state the spin-up and spin-down edge modes

will be slightly shifted with respect to one another on the edge. This will lead to different tunneling matrix elements between the edge and dot for different spin orientations, breaking the channel symmetry in the effective two-channel Kondo model. Thus, the low-energy fixed point will be described by a single-channel Kondo model, rather than the two-channel version. In the case of the non-Abelian  $SU(2)_2$  NAF state, the analysis is nearly identical to the analysis of the anti-Pfaffian state. The edge theories of the two states differ only in the chiralities of the neutral modes, and physics at a single point contact is completely blind to the chirality of the edge modes. Thus, single-channel Kondo behavior is obtained. The edge theory of the Abelian  $K=8$  strong-coupling state is a single chiral boson, which maps to the single-channel Kondo problem (after Toulouse transformation) if the edge-dot repulsive interaction is sufficiently strong (and otherwise flows to the ferromagnetic Kondo fixed point at which the edge and dot decouple, as in all of the models which we consider here).

Finally, the state at  $\nu=12/5$  has another non-Abelian candidate (besides the RR state at  $k=3$ ), a Bonderson and Slingerland (BS) state in the hierarchy built on the Pfaffian. Due to the presence of other edge modes in the BS state at  $\nu=12/5$  (relative to the Pfaffian), we expect the low-energy fixed point to again be described by the single-channel Kondo model. While both the RR and BS state are expected to have the same low-energy fixed point, the crossover at higher energy scales should be governed by the three-channel Kondo model for RR and the two-channel Kondo model for BS. However, due to the unknown intrinsic exchange anisotropies involved, it is likely to be very difficult to conclude much in experiment from this intermediate energy behavior.

### F. Tunneling through a quantum dot

We now consider the situation of bulk  $\nu=2+k/(k+2)$  quantum Hall states on either side of a quantum dot, as shown in Fig. 2. Such a configuration will arise in a two point contact interferometer if the backscattering at the two point contacts is increased until they are both near pinch off (at  $\nu=5/2$ , this will be pinch off of the  $\nu=1/2$  state in the second Landau level while the filled lowest Landau levels are not pinched off). The magnetic field or a side-gate voltage must then be tuned to a degeneracy point of the dot which has been created between the two bulk states.

For the sake of concreteness, we first consider the MR Pfaffian state. The edge Hamiltonian is

$$H_{\text{MR edges}} = \sum_{i=1,2} \int dx \left\{ v_c \frac{2}{4\pi} [\partial_x \phi_i(x)]^2 + iv_n \psi_i \partial_x \psi_i \right\}, \quad (73)$$

where  $i=1,2$  are the two bulk states to the left and right of the dot. The tunneling Hamiltonian takes the form

$$H_{\text{tun}} = \sum_{i=1,2} t_i [d^\dagger \psi_i(0) e^{2i\phi_i(0)} + \psi_i(0) e^{-2i\phi_i(0)} d] + V_i d^\dagger d \partial_x \phi_i(0). \quad (74)$$

This model is tantalizingly close to the four-channel Kondo problem but it is not quite equivalent to it. If we were to

consider the bosonic analogue of this problem—bosons as  $\nu=1$ —then the Hamiltonian would instead be

$$H_{\text{bosonic MR}} = \sum_{i=1,2} \int dx \left\{ v_c \frac{1}{4\pi} [\partial_x \phi_i(x)]^2 + iv_n \psi_i \partial_x \psi_i \right\} + \sum_{i=1,2} t_i [d^\dagger \psi_i(0) e^{i\phi_i(0)} + \psi_i(0) e^{-i\phi_i(0)} d] + V_i d^\dagger d \partial_x \phi_i(0). \quad (75)$$

The second and third lines could then be written as

$$\sum_{i=1,2} t_i (S_+ J_-^i + S_- J_+^i) + V_i S_z J_z^i, \quad (76)$$

where  $J_a^1$  and  $J_a^2$  both generate  $SU(2)_2$  Kac-Moody algebras. For  $t_1=t_2$ ,  $V_1=V_2$ , this is just a cumbersome way of writing the four-channel Kondo Hamiltonian. However, in the  $\nu=1/2$  fermionic version of this problem, which is relevant to us, we need to perform a unitary transformation generated by

$$U = e^{(2-\sqrt{2})id^\dagger d \phi_1(0)}, \quad (77)$$

in order to transform the  $t_1$  term to Kondo form

$$U H U^\dagger = t_1 [d^\dagger \psi_1(0) e^{\sqrt{2}i\phi_1(0)} + \psi_1(0) e^{-\sqrt{2}i\phi_1(0)} d] + t_2 [d^\dagger \psi_2(0) e^{2i\phi_2(0)} e^{-(2-\sqrt{2})i\phi_1(0)} + \psi_2(0) e^{-2i\phi_2(0)} e^{(2-\sqrt{2})i\phi_1(0)} d]. \quad (78)$$

This unitary transformation makes the  $t_2$  term complicated (and highly irrelevant, at least naively). Conversely, we could bring the  $t_2$  term into two-channel Kondo form, at the cost of making the  $t_1$  term complicated. In the case of more general RR states, the same situation appears: we have representations in which the coupling to either bulk state of these terms is simple and of  $k$ -channel Kondo form but we do not have a representation in which both couplings are simple and tractable. It is thus clear that the situation in Fig. 2 does not immediately map to a  $2k$ -channel Kondo model. The nature of its fixed point(s) is an interesting open problem.

In the case of  $t_1 \neq t_2$ , the low-temperature physics can exhibit some interesting crossovers. Suppose that  $t_1 \gg t_2$ , but both are still small (compared to the charging energy of the dot and all other microscopic scales). Then it makes sense to perform the unitary transformation Eq. (77) in order to bring the Hamiltonian to the form (78). In this form,  $t_1$  is relevant and flows to the two-channel Kondo fixed point while  $t_2$  is irrelevant and flows to zero. The most salient feature in this limit will be that the dot does not, as the temperature is lowered, decouple from both bulk QH states but, instead, from only one of them. Consequently, Coulomb-blockade peaks, rather than narrowing as the temperature is lowered, may broaden instead since there will not be a completely isolated dot at  $T=0$ .

### V. DISCUSSION

The intermediate coupling fixed points which we found in this paper may seem somewhat mysterious so we suggest a

physical picture which may help explain them. In order to do so, we draw on the ideas of Ref. 110, in which it was shown that quantum Hall edge states and their interaction with bulk quasiparticles could be understood in terms of boundary conformal field theory as follows. If the boundary of a quantum Hall droplet is treated as a 1D system by “squashing” the edge down and temporarily ignoring the 2D bulk, then the 1D system lives on an interval and, therefore, has conformally invariant boundary conditions at the two ends of the interval (since there was clearly no length scale introduced in the process of reduction to a 1D system). (Here, “droplet” refers to the bulk quantum Hall state, not the “dot.”) Different possible conformally invariant boundary conditions correspond to different possible quasiparticles in the 2D bulk, which is one way in which the system evinces, through bulk-edge interaction, the fact that it is not really 1D. In Abelian states, this is relatively trivial and can usually be ignored but in non-Abelian states, this effect can be significant. In particular, different possible conformally invariant boundary conditions have different boundary entropies, which are given by the quantum dimensions of the associated bulk quasiparticles. Returning now to our new fixed points, we note that the ground-state entropy change associated with our new fixed points is precisely equal to the entropy drop expected if the twofold degeneracy of the dot is lifted and the boundary entropy of the edge excitations of the droplet *increases* by  $\ln d$ , where  $d=2 \cos \pi/(k+2)$  is the quantum dimension of the minimal charge quasiparticle. (Here, we refer to the boundary entropy at the point at which tunneling to the dot occurs when the edge is “squashed” to a 1D interval.) We emphasize that the change in boundary entropy is measured via a change in boundary conditions of the *droplet*/bulk quantum Hall state edge that provides the electrons tunneling to the dot, and so is *independent of the details* of the electronic structure on the dot.

This suggests the following picture for our fixed point, which relies on the fact that the fixed point occurs at intermediate dot-bulk edge coupling so that the charge on the dot fluctuates and is not necessarily integral. (Again, we note this is true independent of the electronic structure of the dot, provided that it is a degenerate two-level system in the absence of tunneling. In particular, it does not matter whether it is in a quantum Hall state or not.) Thus, we can imagine that a quasiparticle-quasihole pair is nucleated so that the quasiparticle is on the droplet and the quasihole is on the dot (or vice versa). This is true, even though at “high energies,” it is electrons that are tunneling, not quasiparticles. The presence of the quasiparticle on the droplet changes the boundary entropy by according to the arguments of Refs. 110 and 111. The presence of the quasihole on the dot lifts the degeneracy between the  $N$ - and  $N+1$ -electron states, thereby changing the boundary entropy by  $-\ln 2$ . To see this, consider, for the sake of concreteness, a Pfaffian state. The energy on a small Pfaffian dot can be modeled by  $E(N)=E_C(n-n_0)^2/2+v_n n/2R$ , where  $n=0,1$  corresponds to  $N$  or  $N+1$  electrons,  $R$  is the length of the edge, and  $E_C$  is the Coulomb charging energy. The first term is the Coulomb energy, with an offset  $0 < n_0 < 1$  while the second term is the energy in the neutral sector (associated with creating a Majorana fermion when an electron is added). By tuning  $n_0=(1+2\frac{v_n}{RE_C})/2$ , we can make

the  $N$  and  $N+1$  electron states degenerate in energy. However, when a charge  $e/4$  quasiparticle is added to the dot, the neutral energy vanishes due to the existence of a dot edge zero mode. Consequently, the degeneracy is lifted. The total entropy drop is thus  $-\ln 2/d$ , as predicted for the multichannel Kondo model.<sup>87,88</sup>

We should emphasize that the precise nature of the electronic states on the quantum dot are not important for our basic result of multichannel Kondo physics in the electron tunneling on and off the dot. In particular, if there is coupling between the edge of the dot and the bulk of the dot, the physics will remain unchanged provided that the energy of other states (not involved in the twofold level degeneracy) is more than the tunnel broadening and  $k_B T$  away in energy. In this case, the level “d” will correspond to some hybridization of edge and bulk states. In fact, the fractional quantum Hall state in the partially filled Landau level in the dot can even be destroyed by finite-size effects (from being too small, for example) and it would not affect our conclusions, provided again that the nearby level spacing is larger than tunnel broadening and temperature. However, there is one detail that is important to our result, and that is that the completely filled lower Landau levels pass freely under the point contact. This detail is important because it means that the tunneling from the bulk quantum Hall edge to the quantum dot only occurs from the “Read-Rezayi” part and not the filled lower Landau levels. If the two integer quantum Hall edges are also backscattered at the point contact, they should also be included in the edge electron tunneling operators (note the plural) to the quantum dot. These additional electron tunneling processes will drive the system towards the single-channel Kondo model at all filling fractions since the “channel isotropy” coming from the parafermion part of the electron operator will no longer protect the multichannel Kondo physics when these additional channels are present. In effect, we expect the physics to be similar to the case of the anti-Pfaffian where the edge electron operator is not unique and the system is described by the single-channel Kondo model at the lowest energies.

There are at least two other scenarios (of which we are aware) where two-channel Kondo physics emerges in the context of quantum Hall states: Fradkin and Sandler discuss a junction of a  $\nu=1/3$  fractional quantum Hall state and a normal metal.<sup>112</sup> Fendley, *et al.* discuss the  $IV$  characteristics of a point contact in a Pfaffian state and show that the crossover from weak to strong backscattering of charge  $e/4$  quasiparticles is described by a variant of the two-channel Kondo model.<sup>113,114</sup> In both cases, the physics is quite distinct from what we discuss in this work. We are unaware of any other works considering multichannel Kondo models derived from the general Read-Rezayi states, or other candidate non-Abelian fractional quantum Hall states.

In order to see multichannel Kondo physics in our setup for the MR state, we need  $V > (2-\sqrt{2})v_c$ . The following is a very crude estimate of the relevant parameters. We take  $v_c \approx 10^5$  m/s (although smaller values are possible if the edge confining potential is smoother), which implies  $\hbar v_c \approx 10^{-10}$  eV m. The coupling  $V$  is the Coulomb energy for an electron on the dot and charge per unit length  $\partial_x \phi_c$  on the edge in the vicinity of  $x=0$ . If  $r$  is the distance between the

dot and the bulk state, and the magnetic length  $\ell_0$  is the short-distance cutoff for edge physics, then we have  $V \propto \frac{e^2}{\epsilon_r} \ell_0$ . For  $r \sim \ell_0$ ,  $V \approx 10^{-10}$  eV m. Thus,  $V$  and  $v_c$  are comparable, and by tuning the edge velocity or the bulk-dot distance, it should be possible to arrange  $V > (2 - \sqrt{2})v_c$ . We emphasize that this estimate is extremely crude because it is difficult to accurately estimate the strength of the interaction  $V$  as it depends on nonuniversal details of the edge, including screening effects from the gates and possibly the edge modes themselves. At any rate, by the same logic, it should be possible to make the exponent  $1/(\frac{V}{\pi v_c} - \frac{1}{2})$  which appears in the Kondo temperature,  $T_K \propto t^{1/(V/\pi v_c - 1/2)}$ , of order 1. If, for instance, it is precisely equal to 1, we will have  $T_K \propto t$ . In edge tunneling experiments,  $t$  values are often in the range  $t/\ell_0 \sim 10$  K, so we expect that the multichannel Kondo fixed point will be observable for accessible temperatures.

A complicating issue in real experiments is possible edge reconstruction, which is expected if the edges are sufficiently smooth.<sup>46-50</sup> When edge reconstruction occurs, pairs of counterpropagating edge modes appear that do not affect the Hall conductance but do affect the edge Hamiltonian. The presence of these modes could in principle destabilize the multichannel Kondo physics we have discussed here at the lowest energies but one generally expects their coupling to the dot to be much weaker due to their greater spatial separation from it. In that case, it may be that even if edge reconstruction is present it will not have much effect over a fairly large range of temperatures and the multichannel Kondo physics will still be observable.

Finally, while the observation of non-Fermi-liquid physics in the thermodynamics of the dot would provide strong evidence for a non-Abelian state of Moore-Read or Read-Rezayi type (at the appropriate filling fraction), the converse is not true: lack of non-Fermi-liquid properties could result from the presence of a particle-hole conjugate state, strong edge reconstruction, or even an Abelian quantum Hall state. Thus, if Fermi-liquid properties are observed, further studies must be done to determine if the state is non-Abelian.

## VI. SUMMARY

In summary, we have shown that a quantum dot coupled via tunneling to a Pfaffian quantum Hall state realizes the channel symmetric two-channel Kondo model while a quantum dot coupled to a Read-Rezayi state of filling factor  $\nu = 2 + k/(k+2)$  leads to a channel symmetric  $k$ -channel Kondo

problem, both without any fine tuning of parameters. These systems will thus exhibit all the known non-Fermi-liquid properties in their thermodynamics, expressed through the charge on the dot, which may be measured capacitively. Because the coupling of a quantum dot to an anti-Pfaffian state generically exhibits Fermi-liquid properties, our results may be used to distinguish between the two leading candidate states for  $\nu=5/2$ : the Pfaffian and the anti-Pfaffian.

Our central results can be understood within the context of the Emery-Kivelson (EK) solution to the two-channel Kondo model. In the EK analysis, a two-channel Kondo model is mapped to a resonant level coupled to some gapless degrees of freedom. In the channel symmetric case, the gapless mode is a Majorana fermion and non-Fermi-liquid impurity physics results. Thus, two-channel Kondo physics is governed by a Majorana resonant level. In the case of channel asymmetry, the level effectively couples to a Dirac fermion and Fermi-liquid impurity physics is found, characteristic of the one-channel Kondo model.

In our work, we reverse the process and ask “to what quantum impurity model does a resonant level coupled to a RR state correspond?” For  $k=2$ , the RR state is the MR Pfaffian state, which has a *single* Majorana mode on its edge. This Majorana mode turns out to play exactly the same role as the Majorana mode in the EK analysis. For the RR states at general  $k > 2$ , the Majorana is replaced by a  $\mathbb{Z}_k$  parafermion (which is a Majorana for  $k=2$ ) which takes the place of the Majorana in the  $k$ -channel Kondo model. The uniqueness of the parafermion mode in the edge of the RR state encodes the physics of channel isotropy in the effective multichannel Kondo model. As a result, the usually fragile multichannel Kondo physics is topologically protected by the non-Abelian quantum Hall state. For particle-hole conjugate states, the low-energy physics is governed by physics other than the channel symmetric multichannel Kondo models. Finally, we find that the transport through a quantum dot between two non-Abelian quantum Hall states is not governed by a  $2k$ -channel Kondo model, but rather by a fixed point that we could not determine with confidence. Its properties are an interesting topic for future study.

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<sup>1</sup>C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, *Rev. Mod. Phys.* **80**, 1083 (2008).

<sup>2</sup>G. Moore and N. Read, *Nucl. Phys. B* **360**, 362 (1991).

<sup>3</sup>M. Greiter, X. G. Wen, and F. Wilczek, *Nucl. Phys. B* **374**, 567 (1992).

<sup>4</sup>C. Nayak and F. Wilczek, *Nucl. Phys. B* **479**, 529 (1996).

<sup>5</sup>R. H. Morf, *Phys. Rev. Lett.* **80**, 1505 (1998).

<sup>6</sup>E. H. Rezayi and F. D. M. Haldane, *Phys. Rev. Lett.* **84**, 4685

(2000).

<sup>7</sup>H. Wang, D. N. Sheng, and F. D. M. Haldane, *Phys. Rev. B* **80**, 241311(R) (2009).

<sup>8</sup>P. Bonderson, A. Feiguin, F. Moller, and J. Slingerland, [arXiv:0901.4965](https://arxiv.org/abs/0901.4965) (unpublished).

<sup>9</sup>A. E. Feiguin, E. Rezayi, K. Yang, C. Nayak, and S. Das Sarma, *Phys. Rev. B* **79**, 115322 (2009).

<sup>10</sup>A. Wójs, *Phys. Rev. B* **80**, 041104(R) (2009).



- <sup>11</sup>M. Dolev, M. Heiblum, V. Umansky, A. Stern, and D. Mahalu, *Nature (London)* **452**, 829 (2008).
- <sup>12</sup>I. P. Radu, J. B. Miller, C. M. Marcus, M. A. Kastner, L. N. Pfeiffer, and K. W. West, *Science* **320**, 899 (2008).
- <sup>13</sup>R. L. Willett, L. N. Pfeiffer, and K. W. West, *Proc. Natl. Acad. Sci. U.S.A.* **106**, 8853 (2009).
- <sup>14</sup>N. Read and E. Rezayi, *Phys. Rev. B* **59**, 8084 (1999).
- <sup>15</sup>P. Bonderson and J. K. Slingerland, *Phys. Rev. B* **78**, 125323 (2008).
- <sup>16</sup>E. H. Rezayi and N. Read, *Phys. Rev. B* **79**, 075306 (2009).
- <sup>17</sup>P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer, New York, 1997).
- <sup>18</sup>A. E. Feiguin, E. Rezayi, C. Nayak, and S. Das Sarma, *Phys. Rev. Lett.* **100**, 166803 (2008).
- <sup>19</sup>R. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, and J. H. English, *Phys. Rev. Lett.* **59**, 1776 (1987).
- <sup>20</sup>W. Pan, J.-S. Xia, V. Shvarts, D. E. Adams, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, *Phys. Rev. Lett.* **83**, 3530 (1999).
- <sup>21</sup>J. S. Xia, W. Pan, C. L. Vicente, E. D. Adams, N. S. Sullivan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, *Phys. Rev. Lett.* **93**, 176809 (2004).
- <sup>22</sup>H. C. Choi, W. Kang, S. Das Sarma, L. N. Pfeiffer, and K. W. West, *Phys. Rev. B* **77**, 081301(R) (2008).
- <sup>23</sup>C. R. Dean, B. A. Piot, P. Hayden, S. Das Sarma, G. Gervais, L. N. Pfeiffer, and K. W. West, *Phys. Rev. Lett.* **100**, 146803 (2008).
- <sup>24</sup>M. Levin, B. I. Halperin, and B. Rosenow, *Phys. Rev. Lett.* **99**, 236806 (2007).
- <sup>25</sup>S.-S. Lee, S. Ryu, C. Nayak, and M. P. A. Fisher, *Phys. Rev. Lett.* **99**, 236807 (2007).
- <sup>26</sup>M. R. Peterson, T. Jolicœur, and S. Das Sarma, *Phys. Rev. B* **78**, 155308 (2008).
- <sup>27</sup>W. Bishara, P. Bonderson, C. Nayak, K. Shtengel, and J. K. Slingerland, *Phys. Rev. B* **80**, 155303 (2009).
- <sup>28</sup>P. Bonderson, A. Kitaev, and K. Shtengel, *Phys. Rev. Lett.* **96**, 016803 (2006).
- <sup>29</sup>P. Bonderson, K. Shtengel, and J. K. Slingerland, *Phys. Rev. Lett.* **97**, 016401 (2006).
- <sup>30</sup>A. Stern and B. I. Halperin, *Phys. Rev. Lett.* **96**, 016802 (2006).
- <sup>31</sup>D. E. Feldman and A. Kitaev, *Phys. Rev. Lett.* **97**, 186803 (2006).
- <sup>32</sup>S. Das Sarma, M. Freedman, and C. Nayak, *Phys. Rev. Lett.* **94**, 166802 (2005).
- <sup>33</sup>L. Fidkowski, [arXiv:0704.3291](https://arxiv.org/abs/0704.3291) (unpublished).
- <sup>34</sup>W. Bishara and C. Nayak, *Phys. Rev. B* **77**, 165302 (2008).
- <sup>35</sup>G. A. Fiete, W. Bishara, and C. Nayak, *Phys. Rev. Lett.* **101**, 176801 (2008).
- <sup>36</sup>X. G. Wen, *Phys. Rev. Lett.* **66**, 802 (1991).
- <sup>37</sup>B. Blok and X. Wen, *Nucl. Phys. B* **374**, 615 (1992).
- <sup>38</sup>B. I. Halperin, *Helv. Phys. Acta* **56**, 75 (1983).
- <sup>39</sup>W. Bishara, G. A. Fiete, and C. Nayak, *Phys. Rev. B* **77**, 241306 (2008).
- <sup>40</sup>D. Berman, N. B. Zhitenev, R. C. Ashoori, and M. Shayegan, *Phys. Rev. Lett.* **82**, 161 (1999).
- <sup>41</sup>D. S. Duncan, C. Livermore, R. M. Westervelt, K. D. Maranowski, and A. C. Gossard, *Appl. Phys. Lett.* **74**, 1045 (1999).
- <sup>42</sup>C. de C. Chamon, D. E. Freed, S. A. Kivelson, S. L. Sondhi, and X. G. Wen, *Phys. Rev. B* **55**, 2331 (1997).
- <sup>43</sup>E. Fradkin, C. Nayak, A. Tselik, and F. Wilczek, *Nucl. Phys. B* **516**, 704 (1998).
- <sup>44</sup>B. Rosenow and B. I. Halperin, *Phys. Rev. Lett.* **98**, 106801 (2007).
- <sup>45</sup>X.-G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford, New York, 2004).
- <sup>46</sup>D. B. Chklovskii, B. I. Shklovskii, and L. I. Glazman, *Phys. Rev. B* **46**, 4026 (1992).
- <sup>47</sup>C. de C. Chamon and X. G. Wen, *Phys. Rev. B* **49**, 8227 (1994).
- <sup>48</sup>X. Wan, K. Yang, and E. H. Rezayi, *Phys. Rev. Lett.* **88**, 056802 (2002).
- <sup>49</sup>Y. N. Joglekar, H. K. Nguyen, and G. Murthy, *Phys. Rev. B* **68**, 035332 (2003).
- <sup>50</sup>D. Orgad and O. Agam, *Phys. Rev. Lett.* **100**, 156802 (2008).
- <sup>51</sup>N. Read, *Phys. Rev. B* **79**, 245304 (2009).
- <sup>52</sup>B. A. Bernevig and F. D. M. Haldane, *Phys. Rev. Lett.* **101**, 246806 (2008).
- <sup>53</sup>S. Jolad and J. K. Jain, *Phys. Rev. Lett.* **102**, 116801 (2009).
- <sup>54</sup>B. A. Bernevig and F. D. M. Haldane, *Phys. Rev. Lett.* **102**, 066802 (2009).
- <sup>55</sup>T. H. Hansson, M. Hermanns, N. Regnault, and S. Viefers, *Phys. Rev. Lett.* **102**, 166805 (2009).
- <sup>56</sup>T. H. Hansson, C.-C. Chang, J. K. Jain, and S. Viefers, *Phys. Rev. B* **76**, 075347 (2007).
- <sup>57</sup>B. I. Halperin, *Phys. Rev. B* **25**, 2185 (1982).
- <sup>58</sup>X. G. Wen, *Phys. Rev. Lett.* **64**, 2206 (1990).
- <sup>59</sup>X. G. Wen, *Phys. Rev. B* **41**, 12838 (1990).
- <sup>60</sup>R. B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).
- <sup>61</sup>X. G. Wen and A. Zee, *Phys. Rev. B* **46**, 2290 (1992).
- <sup>62</sup>X. G. Wen, *Int. J. Mod. Phys. B* **6**, 1711 (1992).
- <sup>63</sup>X.-G. Wen, *Adv. Phys.* **44**, 405 (1995).
- <sup>64</sup>B. I. Halperin, *Phys. Rev. Lett.* **52**, 1583 (1984).
- <sup>65</sup>F. D. M. Haldane, *Phys. Rev. Lett.* **51**, 605 (1983).
- <sup>66</sup>A. H. MacDonald, *Phys. Rev. Lett.* **64**, 220 (1990).
- <sup>67</sup>J. K. Jain, *Phys. Rev. Lett.* **63**, 199 (1989).
- <sup>68</sup>C. L. Kane and M. P. A. Fisher, *Phys. Rev. B* **46**, 15233 (1992).
- <sup>69</sup>C. L. Kane, M. P. A. Fisher, and J. Polchinski, *Phys. Rev. Lett.* **72**, 4129 (1994).
- <sup>70</sup>C. L. Kane and M. P. A. Fisher, *Phys. Rev. B* **51**, 13449 (1995).
- <sup>71</sup>F. D. M. Haldane and E. H. Rezayi, *Phys. Rev. Lett.* **60**, 956 (1988).
- <sup>72</sup>N. Read and D. Green, *Phys. Rev. B* **61**, 10267 (2000).
- <sup>73</sup>X. Wan, K. Yang, and E. H. Rezayi, *Phys. Rev. Lett.* **97**, 256804 (2006).
- <sup>74</sup>X. Wan, Z.-X. Hu, E. H. Rezayi, and K. Yang, *Phys. Rev. B* **77**, 165316 (2008).
- <sup>75</sup>E. Fradkin and L. P. Kadanoff, *Nucl. Phys. B* **170**, 1 (1980).
- <sup>76</sup>V. A. Fateev and A. B. Zamolodchikov, *Sov. Phys. JETP* **62**, 215 (1985).
- <sup>77</sup>D. Karabali and H. J. Schnitzer, *Nucl. Phys. B* **329**, 649 (1990).
- <sup>78</sup>P. Nozieres and A. Blandin, *J. Phys. (Paris)* **41**, 193 (1980).
- <sup>79</sup>G. A. Fiete, G. Zaránd, B. I. Halperin, and Y. Oreg, *Phys. Rev. B* **66**, 024431 (2002).
- <sup>80</sup>D. L. Cox and A. Zawadowski, *Adv. Phys.* **47**, 599 (1998).
- <sup>81</sup>I. Affleck, A. W. W. Ludwig, H.-B. Pang, and D. L. Cox, *Phys. Rev. B* **45**, 7918 (1992).
- <sup>82</sup>V. J. Emery and S. Kivelson, *Phys. Rev. B* **46**, 10812 (1992).
- <sup>83</sup>M. Fabrizio, A. O. Gogolin, and P. Nozières, *Phys. Rev. Lett.* **74**, 4503 (1995).
- <sup>84</sup>N. Andrei and A. Jerez, *Phys. Rev. Lett.* **74**, 4507 (1995).
- <sup>85</sup>A. M. Sengupta and A. Georges, *Phys. Rev. B* **49**, 10020 (1994).

- <sup>86</sup>I. Affleck, *Nucl. Phys. B* **336**, 517 (1990).
- <sup>87</sup>I. Affleck and A. W. W. Ludwig, *Nucl. Phys. B* **360**, 641 (1991).
- <sup>88</sup>I. Affleck and A. W. W. Ludwig, *Nucl. Phys. B* **352**, 849 (1991).
- <sup>89</sup>A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, Cambridge, England, 1998).
- <sup>90</sup>N. Andrei and C. Destri, *Phys. Rev. Lett.* **52**, 364 (1984).
- <sup>91</sup>G. A. Fiete, G. Refael, and M. P. A. Fisher, *Phys. Rev. Lett.* **99**, 166805 (2007).
- <sup>92</sup>R. Ilan, E. Grosfeld, and A. Stern, *Phys. Rev. Lett.* **100**, 086803 (2008).
- <sup>93</sup>R. C. Ashoori, H. L. Stormer, J. S. Weiner, L. N. Pfeiffer, S. J. Pearton, K. W. Baldwin, and K. W. West, *Phys. Rev. Lett.* **68**, 3088 (1992).
- <sup>94</sup>L. DiCarlo, H. J. Lynch, A. C. Johnson, L. I. Childress, K. Crockett, C. M. Marcus, M. P. Hanson, and A. C. Gossard, *Phys. Rev. Lett.* **92**, 226801 (2004).
- <sup>95</sup>A. Furusaki and K. A. Matveev, *Phys. Rev. Lett.* **88**, 226404 (2002).
- <sup>96</sup>K. A. Matveev, *Sov. Phys. JETP* **72**, 1031 (1990).
- <sup>97</sup>K. A. Matveev, *Phys. Rev. B* **51**, 1743 (1995).
- <sup>98</sup>E. H. Kim, Y. B. Kim, and C. Kallin, *J. Phys.: Condens. Matter* **15**, 7047 (2003).
- <sup>99</sup>Y. Oreg and D. Goldhaber-Gordon, *Phys. Rev. Lett.* **90**, 136602 (2003).
- <sup>100</sup>L. Borda and A. Zawadowski, [arXiv:0910.5242](https://arxiv.org/abs/0910.5242) (unpublished).
- <sup>101</sup>A discussion of Klein factors and spin fermionization can be found in A. Kitaev, *Ann. Phys.* **321**, 2 (2006); and in D. G. Clarke, T. Giamarchi, and B. I. Shraiman, *Phys. Rev. B* **48**, 7070 (1993).
- <sup>102</sup>P. Kakashvili and H. Johannesson, *Phys. Rev. Lett.* **91**, 186403 (2003).
- <sup>103</sup>M. Sade, Y. Weiss, M. Goldstein, and R. Berkovits, *Phys. Rev. B* **71**, 153301 (2005).
- <sup>104</sup>P. Wächter, V. Meden, and K. Schönhammer, *Phys. Rev. B* **76**, 125316 (2007).
- <sup>105</sup>I. V. Lerner, V. I. Yudson, and I. V. Yurkevich, *Phys. Rev. Lett.* **100**, 256805 (2008).
- <sup>106</sup>M. Goldstein, Y. Weiss, and R. Berkovits, *EPL* **86**, 67012 (2009).
- <sup>107</sup>M. Goldstein and R. Berkovits, [arXiv:0907.0424](https://arxiv.org/abs/0907.0424) (unpublished).
- <sup>108</sup>J. B. Miller, I. P. Radu, D. M. Zumbühl, E. M. Levenson-Falk, M. A. Kastner, C. M. Marcus, L. N. Pfeiffer, and K. W. West, *Nat. Phys.* **3**, 561 (2007).
- <sup>109</sup>R. M. Potok, I. Rau, H. Shtrikman, Y. Oreg, and D. Goldhaber-Gordon, *Nature (London)* **446**, 167 (2007).
- <sup>110</sup>P. Fendley, M. P. A. Fisher, and C. Nayak, *Ann. Phys.* **324**, 1547 (2009).
- <sup>111</sup>P. Fendley, M. Fisher, and C. Nayak, *J. Stat. Phys.* **126**, 1111 (2007).
- <sup>112</sup>N. P. Sandler and E. Fradkin, *Phys. Rev. B* **63**, 235301 (2001).
- <sup>113</sup>P. Fendley, M. P. A. Fisher, and C. Nayak, *Phys. Rev. Lett.* **97**, 036801 (2006).
- <sup>114</sup>P. Fendley, M. P. A. Fisher, and C. Nayak, *Phys. Rev. B* **75**, 045317 (2007).